VOL. L. SEPTEMBER, 1924. NO. 7.

REPORTS, PAPERS, DISCUSSIONS, AND MEMOIRS

This Society is not responsible for any statement made or opinion expressed in its publications.

CONTENTS

PLATES

For Index to all Papers, the discussion of which is current in *Proceedings*, see the second page of the cover.

SECONDARY STRESSES IN BRIDGES*

By Cecil Vivian von Abo,† Jun. Am. Soc. C. E.

Synopsis

The paper first describes all the methods which have been proposed for the complete analysis of secondary stresses due to the rigidity of the joints in bridge trusses. Approximate methods, however, such as those of Engesser and Landsberg, which neglect the effect of the web system, are not included.

A typical through railway bridge of six panels is analyzed by each method in turn, using unsymmetrical loading, in order to test more thoroughly the advantages and disadvantages of each method. A critical comparison is then made of the methods thus applied.

Influence lines are developed for the secondary stresses in all the members of the main trusses, in the top chords as effected by the top lateral system, and in a typical cross-frame. These influence lines are then combined for the purpose of computing the maximum secondary stresses in certain typical members.

PART I

1.—Introduction

Prior to 1879, no form of mathematical analysis was known, and no empirical results were available, which would enable an engineer to calculate with reasonable accuracy the stresses in a structure as built. Even after that date, when Manderla gave his excellent solution of secondary stresses due to the restraint of members meeting at a joint, little attention was paid to such stresses for many years. The labor involved in the calculations was usually the reason given for this neglect; but it is possible that lack of familiarity with the subject played an equally important part.

In recent years the construction of such phenomenal bridges as the Quebec cantilever, the Sciotoville continuous truss, and the Hell Gate arch, has awakened new interest in the subject. The Engineering Profession owes much to those in charge of these structures for the accounts of their investigations. Of particular interest is the paper by D. B. Steinman, M. Am. Soc. C. E., on "Stress Measurements on the Hell Gate Arch Bridge" (33),‡ with the full discussion by leading bridge engineers.

Note.—Written discussion on this paper will be closed with the January, 1925, *Proceedings.* When finally closed, the paper, with discussion, will be published in *Transactions.*

* This paper was presented by Mr. von Abo in September, 1922, as a thesis to the Committee on Graduate Studies of McGill University for the degree of Doctor of Philosophy.

† Parys, Orange Free State, South Africa.

‡ Figures in parentheses refer to numbered articles in the Bibliography.

In recent specifications the tendency is to recognize the importance of secondary stresses. Higher unit stresses are properly permitted for combinations of secondary and primary stresses. This undoubtedly leads to a better distribution of metal, and to a more uniform strength throughout the structure. Although many think that the calculation of secondary stresses in ordinary spans will seldom be necessary, the writer believes that if the Engineering Profession at large becomes thoroughly versed in the matter, the strength and economy of all bridges will be superior to any that may be built with what has been so appropriately called "the factor of ignorance".

Definition of Secondary Stresses.—The most general definition of "secondary stress" would include shearing and torsional stresses. For the purposes of this paper, however, these are not considered, but rather the direct stresses, tensile or compressive, of the fibers parallel to the center line of a member. This member, under loading, transmits a certain total stress, which, if divided by the area of cross-section, gives an average unit stress; this is the uniform unit stress in the member of the ideal bridge. If, however, in the actual case, some particular fiber be considered and it were possible to determine its elongation, and, therefore, its unit stress, in general this unit stress would differ from the average. Therefore, at any particular cross-section of a member in a bridge, the amount by which the unit stress in any longitudinal fiber differs from the average unit stress, found by dividing the total direct stress by the area, is called the secondary stress of that fiber.

This paper is concerned with the maximum variation, at the four edges of the member, as far away from the center line as possible, for it is there that stresses are most severe.

Classification.—In 1914 a committee of the American Railway Engineering Association classified the secondary stresses in bridges.* The writer, however, suggests the following classification:

(1).—Secondary stresses, arising from rigid or partly rigid connections.

(2).—Secondary stresses, arising from the beam action of members.

(3).—Secondary stresses developed in axially loaded members.

(4).—Secondary stresses due to faulty design.

Class (1) would take care of any secondary stresses that may arise in the various plane frames due to the rigidity of the joints in those planes. The bridge trusses, the top lateral system, the floor system, and the various cross-frames would contribute to this class. Imagine a bridge to be in equilibrium under a stationary train load. Certain secondary stresses will be found in each of the trusses. The top chords of both trusses change in length, while the other top lateral members remain constant in length; hence, secondary stresses are found in the top lateral system. In the same way, the changes in length of the bottom chords, with no changes in length of the floor-beams and stringers, introduce secondary stresses in the floor system. Finally, in the cross-frames the floor-beams bend under loads in the adjoining panels; if these floor-beams were pin-connected to the two trusses, the bending stresses in the floor-beams would be simply those of Class (2), but, as the floor-beams are rigid

* *Proceedings*, Am. Ry. Eng. Assoc., Vol. 15 (1914).

parts of the cross-frames, secondary stresses exist in the frames as a whole, which should be regarded, therefore, as in Class (1).

Imagine that a gale is blowing. It has the effect of changing this arrangement, which is symmetrical about the vertical plane dividing the bridge in half along its length, making it unsymmetrical, while certain direct stresses induced by the wind in the top lateral and the floor systems superimpose secondary stresses on the ones already found. In the case of a wind, too, the end portals come into play and are placed in Class (1).

Class (2) needs little explanation, for a member has bending stresses developed by its own weight and by any wind pressure against it.

Class (3) is seldom, if ever, reckoned as part of a study of secondary stresses. In leading up to the definition of secondary stresses, the assumption was made that the compression members in the ideal bridge were to behave just opposite to those in tension. Euler showed, however, that a column does not transmit a uniform compression under an axial load. Since then many investigators have striven after solutions of this problem, which is all the more complex when the column is a built-up one with diaphragms, latticing, etc. It must be remembered that this member suffers bending under transverse loads, such as dead weight or wind; any axial load, which may now be regarded as an eccentric load, induces further bending stresses.

Mathematical analysis, supported by practical investigation, shows that, in the case of eye-bars transmitting an axial load, the fibers across the section through the center of the pin do not suffer uniform deformation. In fact, the unit stress of the fibers on the circumference of the hole is three times the average stress across the section. This property of an eye-bar is, therefore, a cause of secondary stresses.

Under Class (4) appear eccentric connections. A careful set of measurements would indicate the values of these eccentricities fairly accurately, in which case they could be judiciously incorporated in analyses of Class (1). There is in this class, however, a condition which defies the engineer, such as occurs, for example, when a compression member is made up of two channels which are supposed to have the same cross-section and to share the load equally. Generally, such an equal distribution is not found. This, of course, would cause a variation in the secondary stress from that found by assuming the load to be shared equally.

The Superposition Theory, Applied to Secondary Stress Analysis.—In analyzing for the primary stresses in a structure, that is, regarding it as ideal and the stresses as pure tensions or pure compressions, it is possible to solve the problem in three dimensions, but it is far easier to regard the structure as being made up of a number of planes, solving for the stresses in each plane produced by forces that act in that plane, and simply adding algebraically the stresses for all the planes of which the member forms a part. This is the superposition theorem—the method of analysis commonly used by engineers.

In analyzing for secondary stresses, the only known method is to solve in a plane structure; even if it were possible to solve for secondary stresses in a space frame, the case of the primary stresses is conclusive evidence that the corresponding secondary stress problem will be most complicated and laborious.

Suppose for the moment that, by some method or other, the secondary stresses have been determined in the members belonging to one of the plane frameworks, for instance, one of the trusses, due to a certain loading in that plane. The so-called neutral axes of the members are really neutral planes. These planes pass through the centers of gravity of the members and are perpendicular to the plane of the truss. The fibers, the secondary stresses of which presumably have been found, are those lying along the planes on both sides of, parallel to, and farthest remote from, the neutral planes.

This same loading also induces secondary stresses in the top lateral system, but in this case the plane is horizontal, and the neutral planes are vertical. (This assumes that the top chords are straight, but the argument applies to the case where the top chords are curved or made up of a number of straight parts.) The top chords of the trusses are also parts of this new plane. In the former case, the neutral planes were planes perpendicular to the vertical truss plane; in the latter, the neutral planes are perpendicular to the horizontal top lateral plane. Whereas the secondary stresses of the top chord, as part of the truss, were for the fibers that lie along its top and bottom sides, the secondary stresses of the top chord, as part of the top lateral system, are for the vertical sides. The four corner fibers are affected in both analyses and, in order to obtain the greatest variations from the average stress, it is necessary to compound the secondary stresses in the two cases. This overlapping effect should be kept in mind throughout the analysis of a bridge, in order to arrive at a true estimate of the secondary stresses.

Another Definition of Secondary Stress.—A good practical definition of these stresses is as follows:

"The secondary stress in any member at any section is equal to one-half the difference between the two extreme fiber stresses measured at the same section; the primary stress is then equal to one-half the sum of the same extreme fiber stresses."

This is the case where the member is part of a single plane framework—no other planes influence it. The two extreme fiber stresses are those belonging to the two sets of fibers on opposite sides of the neutral plane. This definition is justified only if all the stresses are within the elastic limit and if Bernoulli's assumption holds. Let *ABCD* (Fig. 1), be any cross-section of the member; the definition is true if the fibers on one side of the neutral plane, *XX*, and at equal distances from it are stressed equally. That is, *ab* represents the stress in every longitudinal fiber passing through *AB*, and *cd* represents the stress in every longitudinal fiber passing through *CD*.

Therefore, the cross-section, *ABCD*, may be replaced by any line parallel to *AD* and *BC*, and the problem regarded as an investigation of the stresses along this line. Hence the secondary stress is $\frac{1}{2}(ab - cd)$ and the primary stress $\frac{1}{2}(ab + cd)$.

It is somewhat different in practice, however. Following the assumption of Bernoulli, the diagram will be shown in Fig. 2. The trapezoid, *abb′a′*, repre-

sents the stresses along the fibers that cross AB, $bb'c'c$ those along the fibers that cross BC, etc. In the case considered, $abb'a'$ and $cdd'c'$ were rectangles. Assuming that the variations along AB, BC, CD, and DA are uniform, it appears that, as the bb' of $abb'a'$ is the bb' of $bb'c'c$, etc., the primary stress is $\frac{1}{4}(aa' + bb' + cc' + dd')$, and the secondary stress at A is $\frac{1}{4}(aa' + bb' + cc' + dd') - aa'$, with similar expressions at B, C, and D.

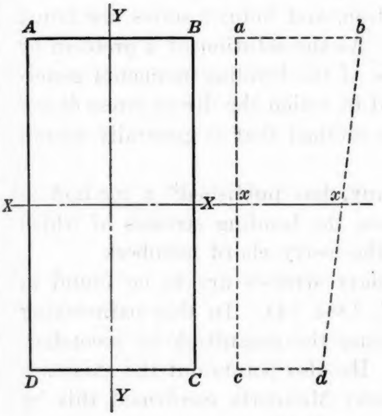


FIG. 1.

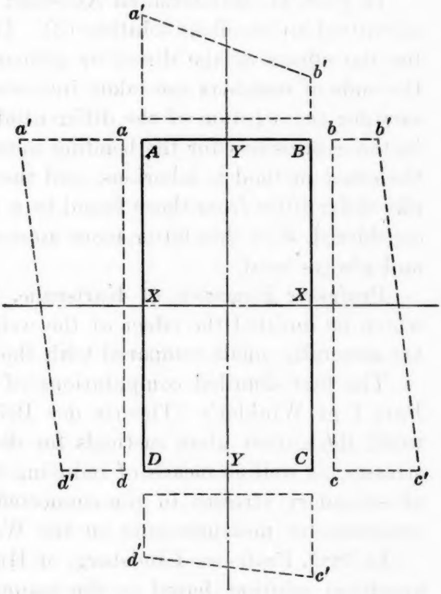


FIG. 2.

This affords a more comprehensive definition:

"The primary stress is the average of the actual stresses found at the four corners of a member; the secondary stress is then the difference between this average and the actual stress of the fiber considered."

2.—Historical Notes

Derivation of Names.—The term "secondary stresses" is due to German writers, who classed all stresses in a bridge under (1) *Grundspannungen* (primary stresses), and (2) *Sekundarspannungen* (secondary stresses).

The *Grundspannungen* are themselves divided into (*a*) *Hauptspannungen* (principal stresses), and (*b*) *Zusatzspannungen* (additional stresses). The *Hauptspannungen* arise from the dead load and the live load, the latter being considered as a static load, while the *Zusatzspannungen* are due to wind pressure, traction, centrifugal force, and impact. The *Grundspannungen* are the stresses of the ideal structure, all pure tensions and pure compressions.

The fixedness of the joints, the weights of the members (distributed and not concentrated at the ends), the wind pressure acting in the same way, the property of axially loaded columns—all contribute to the distortion of the

uniform stress distribution of the ideal case, the amount of distortion for a fiber being its *Sekundarspannung* (secondary stress).

Account of the Theoretical Investigations for Bridge Trusses.—In 1877, the Technical University of Munich offered a prize for the solution of a problem, worded by Professor Asimont as follows: What stresses arise in the members of a bridge truss owing to the fact that the angles of the truss triangles suffer no change?

In 1879, H. Manderla, an Assistant in the Technical University of Munich, submitted an excellent solution (3). His mathematical analysis is most exact, for the effects of the direct or primary stresses on the bending moments at the ends of members are taken into account. Hyperbolic functions are necessary for the solution of the differential equation, and infinite series are found in the expressions for the bending moments. As the solution of a problem by the exact method is laborious, and the values of the bending moments generally differ little from those found by a method in which the direct stress is not considered, it is this latter more approximate method that is generally known and always used.

Professor Engesser, of Karlsruhe, Germany, has published* a method in which he omitted the effect of the web system, the bending stresses of which are generally small compared with those in the heavy chord members.

The first detailed computations of secondary stresses are to be found in Part I of Winkler's "Theorie der Brücken", 1881 (4). In this painstaking work, the author gives methods for determining the magnitude of secondary stresses, as well as means of reducing them. He also points out the existence of secondary stresses in pin-connected trusses; Manderla confirmed this by extensometer measurements on the Waltenhofen Bridge.

In 1885, Professor Landsberg, of Hannover, Germany, gave an approximate graphical solution, based on the assumption that only the chords are riveted (5). This, like Engesser's method, neglects the web system. During the same year, Professor Müller-Breslau, of Berlin, Germany, contributed an analytical solution (6). In 1890, Professor W. Ritter, of Zurich, Switzerland, gave his graphical solution (7), and, in 1892, Professor O. Mohr presented his semi-graphical one. In 1892, Professor Engesser published a thorough treatise on secondary stresses (8).

A remarkable work on secondary stresses was produced in 1902 by Professor E. Patton, of Moscow, Russia (10). His aim was to prove the great advantages of the double-intersection trusses, then in common use in Russia. Professor Patton made a summary of the results obtained by computing the maximum primary and secondary stresses of five different bridge trusses, together with those obtained by other prominent scientists. These data enabled him to draw certain curves and study the relations between secondary stresses and the stiffness of the members. The Patton curves, however, give only a rough approximation, as the effect of the truss system and of the distribution of the loads on secondary stresses had not been taken into consideration.

It was not until 1905 that a written discussion in English appeared. In that year, Isami Hiroi, M. Am. Soc. C. E., of the Tokyo Imperial University,

* *Zeitschrift für Baukunde*, 1879.

Tokyo, Japan, published his "Statically-Indeterminate Stresses in Frames Commonly Used for Bridges" (12). He uses the method of least work to find the secondary stresses in bridge trusses and also in the cross-frames. The method is long, and has no checks; therefore, it is not often used. The German periodical, *Der Eisenbau* contains some remarkable treatises on the subject of secondary stresses, including a method of solving secondary stresses in various types of frames by Professor Mohr, of Dresden, Germany. He makes use of what is known as elastic weights, a method purely analytical (19).

A second method found in *Der Eisenbau* is due to Dr. Pirlet, of Aachen (20). He treats pairs of bending moments at the ends of adjacent members entering a joint as the unknown quantities; there are as many such pairs as there are angles in the truss, or 3 (n-2), in which n is the number of joints and (n-2) the number of triangles. He obtains equations that are of Gauss' normal type. His method is interesting and shows how closely the solutions of secondary stresses and of indeterminate stresses are related. The practical use, however, is small, as Manderla's method requires the solution of only n simultaneous equations of Gauss' normal type, and, therefore, is far superior to that of Pirlet.

Turneaure has developed invaluable methods of tabulating the various quantities, of solving the simultaneous equations, and especially of choosing the panel-point loads for influence lines (25).

In 1913, Mr. José Páez submitted a remarkably fine thesis to Cornell University, entitled "A Comparative and Critical Study of the Different Methods of Computing Secondary Stresses in Bridge Trusses" (27). In 1917, Dr. T. E. Mao also presented a thesis to Cornell University on the study of a large spandrel-braced arch (28); he made extensive use of Gauss' method to solve the equations leading up to the secondary stresses. Two years later, he presented a thesis on "Secondary Stresses" (34) to the Carnegie Institute of Technology, Pittsburgh, Pa., when he obtained the degree of Doctor of Engineering.

Practical Investigations for Bridge Trusses.—The foregoing is a brief account of the theoretical investigations of secondary stresses in bridge trusses due to the rigidity of the joints. Turning to practice for verification, it is most gratifying to know that many investigations in the field have been made. Manderla was the first investigator; he experimented on the Waltenhofen Bridge. In 1883, Frankel made the first direct measurement of secondary stresses. In 1899, Mesnager published an account of the measurement of stresses in a Pratt truss of 180-ft. span on the Orleans Railway in France.* In 1901, M. Rabut described a series of stress measurements which had been made on a bridge of the Orleans Railway. During 1905 and 1906, W. Gehler conducted a series of tests on a railway bridge, of 128-ft. span, near Elsterwerda, Saxony. In addition to extensometers of the Frankel type, he used sensitive levers. A detailed report of the computations, experiments, and results is to be found in Professor Gehler's book (15). During 1907, 1908, and 1909, extensive secondary stress measurements were made by Professors Crandall and

* "Les fatigues réelles et fatigues calculées dans un pont à grandes mailles", *Annales des Ponts et Chaussées*, 1899.

Turneaure, under the auspices of the American Railway Engineering Association. This investigation was primarily to determine the "additional" stress that had to be allowed for impact. A number of plate girders and truss bridges, ranging in span from 30 to 440 ft., were tested for both impact and secondary stresses, but in the latter case the investigations were performed only when the test load was traveling over the bridge at speeds that produced no impact.

In 1915, Mr. Steinman carried out extensive measurements on the Hell Gate Arch (33). Some results were found to conflict with theory, but on the whole the investigation is a remarkable verification of the theory. Recently, quite a number of investigations have been carried out in Switzerland and in England (36). The results of many investigations have not been published, and, therefore, are not available to the profession.

Cross-Frames.—Dr. Hiroi explains the method of solution of a cross-frame using the principle of least work (12). Dr. Cyril Batho, of McGill University, has a neat way of applying the slope-deflection method in his own solution. Professor Mohr makes use of elastic weights, not only to solve problems of cross-frames, but also the Vierendeel truss, a type found in Europe, principally in Belgium; this is a truss that has no diagonals, but depends on the stiffness of the chords to transmit the shear.*

Column Action.—A history of secondary stress investigations must include an account of the investigations on columns. Such an account would practically mean a résumé of the theory of columns, but only two investigations will be mentioned. The treatise by H. M. Westergaard, Assoc. M. Am. Soc. C. E., on "Buckling of Elastic Structures"(35), not only throws much light on the subject but also has a useful bibliography. Perhaps the finest practical contribution was the series of investigations on columns and eyebars, carried out under the direction of H. M. MacKay, M. Am. Soc. C. E., of McGill University, in connection with the Quebec Cantilever Bridge.†

PART II.—ANALYSIS OF SECONDARY STRESSES IN A BRIDGE TRUSS DUE TO THE RIGIDITY OF ITS JOINTS

1.—Manderla's Method

Introduction.—Consider the analysis of a truss made up of triangular elements. If quadrilaterals or other polygons are involved, as in the case of trusses with subdivided panels, the introduction of imaginary members, with zero cross-section and zero load, transforms such polygons into triangles. The assumption is that, if there are two trusses, identical except that one has riveted joints while the other has perfectly frictionless hinges, and if these trusses could be interchanged, the positions assumed by the joints of the riveted truss under a certain load and those by the corresponding joints of the hinged truss under the same load are identical.

When a truss with frictionless hinges is loaded every member suffers some definite elongation or contraction, and the positions of the joints change to

* Professor Mohr's method may be found in *Der Eisenbau*, III, 1912, p. 85.

† Report of Board of Engineers, Quebec Bridge; also, R. Mayer, "Die Knickfestigkeit," Berlin, 1921.

agree with these deformations. The important facts to remember are that (1) all the members remain perfectly straight; (2) no bending stresses are introduced; (3) the angles of each triangular element are changed; and (4) the angles between adjacent members at a joint are likewise changed.

If the joints were fixed, and some means were used for forcing the members to positions where the angles at the joints again assume the values they had before loading, the members of the hinged truss would take up the exact positions and shape of the corresponding members in the riveted truss under the same load, for the effect of a gusset-plate at a joint is to keep the angle between any pair of members at the joint constant at all times. The problem, therefore, is to calculate the changes of angle between adjacent members of the hinged truss, and also the value of the bending moment at each end of every member necessary to make these changes zero.

Changes of Angle in a Triangular Element.—Knowing the unit axial stress in each member of a given triangular element, as well as the modulus of elasticity of the material, the angular changes may be computed by any one of several methods. The following is a neat graphical method due to Ritter. Let ABC (Fig. 3) be any triangular truss element and s_1, s_2, and s_3, the unit stress in the sides, BC, CA, and AB, respectively.

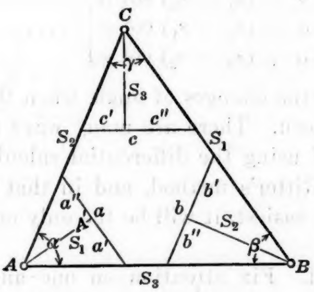


FIG. 3.

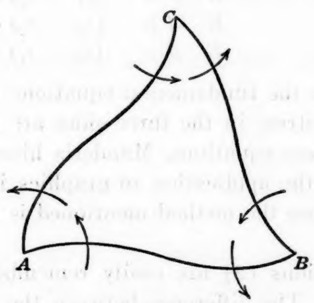


FIG. 4.

Draw a line through A perpendicular to the opposite side, BC, and cut off a length, AA', equal to s_1, the stress, BC. Through A' draw a line parallel to BC and terminated by the sides, BC, AB. Repeat this operation for the other two vertices.

For a change of length in the member BC, the contributions toward the changes of angle will be found to be:

$$E \,.\, \Delta\,\alpha_1 = +\,a$$
$$E \,.\, \Delta\,\beta_1 = -\,a''$$
$$E \,.\, \Delta\,\gamma_1 = -\,a'$$

for a change in the member, CA:

$$E \,.\, \Delta\,\alpha_2 = +\,b$$
$$E \,.\, \Delta\,\beta_2 = b'$$
$$E \,.\, \Delta\,\gamma_2 = -b'',$$

and, finally, for a change in the member, AB:

$$E \,.\, \varDelta\, \alpha_3 = -c''$$
$$E \,.\, \varDelta\, \beta_3 = -c'$$
$$E \,.\, \varDelta\, \gamma_3 = +c.$$

The total change, $\varDelta\, \alpha$, of α due to a simultaneous change in length in all the members will be found by compounding the three contributions to α in the previous equations, namely:

$$\left.\begin{array}{ll} & E \,.\, \varDelta\, \alpha = E \,.\, \varDelta\, \alpha_1 + E \,.\, \varDelta\, \alpha_2 + E \,.\, \varDelta\, \alpha_3 \\ & \qquad = +a - b' - c'' \\ \text{Similarly,} & \\ & E \,.\, \varDelta\, \beta = +b - c' - a'' \\ \text{and,} & \\ & E \,.\, \varDelta\, \gamma = +c - a' - b'' \end{array}\right\} \ldots\ldots\ldots\ldots (1)$$

To make these results suitable for analytical computation it is to be noted that:

$$\begin{array}{lll} a' = s_1 \cot \beta & b' = s_2 \cot \gamma & c' = s_3 \cot \alpha \\ a'' = s_1 \cot \gamma & b'' = s_2 \cot \alpha & c'' = s_3 \cot \beta \\ a = s_1 (\cot \beta + \cot \gamma) & b = s_2 (\cot \gamma + \cot \alpha) & c = s_3 (\cot \alpha + \cot \beta) \end{array}$$

Therefore, Equations (1) become:

$$\left.\begin{array}{l} E \,.\, \varDelta\, \alpha = (s_1 - s_2) \cot \gamma + (s_1 - s_3) \cot \beta \\ E \,.\, \varDelta\, \beta = (s_2 - s_3) \cot \alpha + (s_2 - s_1) \cot \gamma \\ E \,.\, \varDelta\, \gamma = (s_3 - s_1) \cot \alpha + (s_3 - s_2) \cot \alpha \end{array}\right\} \ldots\ldots\ldots\ldots (2)$$

These are the fundamental equations for the changes of angle when the intensities of stress in the three sides are known. There are many ways of arriving at these equations, Manderla himself using the differential calculus. Because of the application to graphics in Ritter's method, and in that of Mao, and because the method mentioned is the easiest, it will be the only one reproduced.

Equations (2) are easily remembered. Fix attention on one angle of a triangle. The difference between the stresses in the opposite side and one of the adjacent sides is multiplied by the co-tangent of the included angle. Two such terms are found, their sum being the product of the required change in the angle considered and E, the modulus of elasticity.

Bending Moments Introduced at the Joints.—Consider the triangle, ABC (Fig. 4), representing the shape that the frame of Fig. 3 would assume if the members were forced round so that the angles at A, B, and C have the same value as before loading. Fig. 4 also represents the shape that the triangular framework of the riveted truss actually assumes under load. The member, AB, is represented by itself in Fig. 5.

Let the bending moments at A and B be M_1 and M_2, both counterclockwise. There must be shearing forces at A and B; let these be V_1 upward at A and V_2 downward at B. They are, of course, equal in magnitude, for AB is in equilibrium and V_1 and V_2 are the only two vertical forces.

Finally, there is some force, P, along AB—this is the total stress transmitted by the member and assumed to be tension in this case.

The moment at any point, N, at a distance, x, from A is:

$$M_x = M_1 - P \cdot y - V_1 \cdot x \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

in which y is the offset of the center of gravity of Section N from AB.

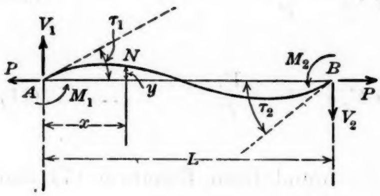


FIG. 5.

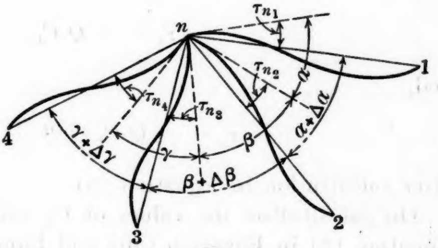


FIG. 6.

The differential equation of the elastic line is:

$$\frac{d^2 y}{d x^2} = -\frac{M_x}{E I} = -\frac{M_1}{E I} + \frac{P \cdot y}{E I} + \frac{V_1 \cdot x}{E I} \ldots\ldots\ldots\ldots (4)$$

in which, I is the moment of inertia of the member. If the primary stress, P, is compression, the only change in Equation (4) is $-\dfrac{P \cdot y}{E I}$ instead of $+\dfrac{P \cdot y}{E I}$.

Let $\dfrac{P}{E I} = Q^2$, which is, therefore, a constant as long as the condition of loading remains.

Equation (4) now becomes:

$$\frac{d^2 y}{d x^2} - Q^2 \cdot y = -\frac{M_1}{E I} + \frac{V_1 \cdot x}{E I} \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

This is a standard differential equation, solved by finding (a) its complementary function by assuming $y = e^{mx}$; and (b) its particular integral by inspection.

The solution is:

$$y = C_1 \cdot e^{+Qx} + C_2 \cdot e^{-Qx} + \frac{M_1}{P} - \frac{V_1 x}{P} \ldots\ldots\ldots\ldots (6)$$

in which, C_1 and C_2 are constants to be determined from the end conditions. As $y = 0$ both when $x = 0$ and when $x = L$,

$$C_1 = -\frac{M_1}{P} \cdot \frac{(1 - e^{-QL})}{(e^{+QL} - e^{-QL})} + \frac{V_1 L}{P} \cdot \frac{1}{(e^{+QL} - e^{-QL})} \ldots\ldots (7)$$

and,

$$C_2 = +\frac{M_1}{P} \cdot \frac{(1 - e^{-QL})}{(e^{+QL} - e^{-QL})} - \frac{V_1 L}{P} \cdot \frac{1}{(e^{+QL} - e^{-QL})} \ldots\ldots (8)$$

The slope of the elastic line at any point, N, is:

$$\frac{d y}{d x} = +Q\, C_1\, e^{+Qx} - Q\, C_2\, e^{-Qx} - \frac{V_1}{P} \ldots\ldots\ldots\ldots (9)$$

but,

$$\tau_1 = \frac{d\,y}{d\,x} \text{ for } x = 0, \text{ and } \tau_2 = \frac{d\,y}{d\,x} \text{ for } x = L$$

hence,

$$\tau_1 = +\,Q\,C_1 - Q\,C_2 - \frac{V_1}{P} \qquad (10)$$

and,

$$\tau_2 = +\,Q\,C_1\,e^{+QL} + Q\,C_2\,e^{-QL} - \frac{V_1}{P} \qquad (11)$$

after substitution in Equation (9).

On substituting the values of C_1 and C_2 found from Equation (7) and Equation (8) in Equation (10) and Equation (11):

$$\tau_1 = \frac{M_1}{P} \cdot Q \cdot \frac{(e^{+QL} - 1)}{(e^{+QL} + 1)} + \frac{V_1}{P}\left(\frac{2}{e^{+QL} - e^{-QL}} \cdot QL - 1\right) \qquad (12)$$

and

$$\tau_2 = \frac{M_1}{P} \cdot Q \cdot \frac{(e^{+QL} - 1)}{(e^{+QL} + 1)} + \frac{V_1}{P}\left(\frac{e^{+QL} + e^{-QL}}{e^{+QL} - e^{-QL}} \cdot Q\,L - 1\right) \qquad (13)$$

Making use of hyperbolic functions, these equations become:

$$\tau_1 = \frac{M_1}{P} \cdot Q \tanh \cdot \frac{Q\,L}{2} + \frac{V_1}{P}\left(\frac{1}{\sinh \cdot Q\,L} \cdot Q\,L - 1\right) \qquad (14)$$

$$\tau_2 = \frac{M_1}{P} \cdot Q \tanh \cdot \frac{Q\,L}{2} + \frac{V_1}{P}\left(\frac{\cosh \cdot Q\,L}{\sinh \cdot Q\,L} \cdot Q\,L - 1\right) \qquad (15)$$

Eliminating V_1 from Equation (14) and Equation (15), a relation is found between M_1 and the deflection angles, τ_1 and τ_2. This is:

$$M_1 = \frac{2\,E\,I}{L}\,(2\,a\,\tau_1 + b\,\tau_2) \qquad (16)$$

in which,

$$a = \frac{1}{8}\left(\frac{Q^2\,L^2}{Q\,L \coth \cdot \dfrac{Q\,L}{2} - 2} + Q\,L \cdot \coth \cdot \frac{Q\,L}{2}\right)$$

and,

$$b = \frac{1}{4}\left(\frac{Q^2\,L^2}{Q\;L \coth \cdot \dfrac{Q\,L}{2} - 2} - Q\,L \cdot \coth \cdot \frac{Q\,L}{2}\right)$$

Replacing M_1 by M_2 and interchanging τ_1 and τ_2,

$$M_2 = \frac{2\,E\,I}{L}\,(2\,a\,\tau_2 + b\,\tau_1) \qquad (17)$$

These values of a and b may be expressed in the form of infinite series by the use of the known series:

$$\coth \cdot x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2\,x^5}{945}, \text{ etc.,} \qquad (18)$$

Thus,

$$a = 1 + \frac{(Q\,L)^2}{30} - \frac{11\,(Q\,L)^4}{25\,000} +, \text{etc.}, \ldots\ldots\ldots\ldots (19)$$

$$b = 1 - \frac{(Q\,L)^2}{60} + \frac{13\,(Q\,L)^4}{25\,000} -, \text{etc.}, \ldots\ldots\ldots\ldots (20)$$

Equation (19) and Equation (20) are for the case of P as tension. If P is compression, Q^2 is negative, and the signs of the terms in the even places in Equation (19) and Equation (20) must be changed, as follows:

$$a = 1 - \frac{(Q\,L)^2}{30} - \frac{11\,(Q\,L)^4}{25\,000}, \text{etc.}, \ldots\ldots\ldots\ldots (21)$$

$$b = 1 + \frac{(Q\,L)^2}{60} + \frac{13\,(Q\,L)^4}{25\,000}, \text{etc.}, \ldots\ldots\ldots\ldots (22)$$

Summarizing, the fundamental deflection formulas are:

$$\left.\begin{aligned} M_{12} &= \frac{2\,E\,I}{L}\,(2\,a\,\tau_{12} + b\,\tau_{21}) \\ M_{21} &= \frac{2\,E\,I}{L}\,(2\,a\,\tau_{21} + b\,\tau_{12}) \end{aligned}\right\} \ldots\ldots\ldots\ldots (23)$$

in which,

$$\left.\begin{aligned} a &= 1 \pm \frac{(Q\,L)^2}{30} - \frac{11\,(Q\,L)^4}{25\,000} \pm \cdots \\ b &= 1 \mp \frac{(Q\,L)^2}{60} + \frac{13\,(Q\,L)^4}{25\,000} \mp \cdots \end{aligned}\right\} \ldots\ldots\ldots\ldots (24)$$

the upper signs in Equation (24) being used when P is tension, and the lower when P is compression.

A slight change in the notation has been introduced. The member, AB, is called 12, and the bending moment and deflection angle at End 1 are denoted by M_{12} and τ_{12}, respectively, while those of End 2 are M_{21} and τ_{21}, respectively.

Expressing τ_1 and τ_2 in terms of M_{12} and M_{21}, Equation (23) will give:

$$\left.\begin{aligned} \tau_{12} &= \frac{L}{6\,E\,I}\,(2\,c\,M_{12} - d\,M_{21}) \\ &\text{and,} \\ \tau_{21} &= \frac{L}{6\,E\,I}\,(2\,c\,M_{21} - d\,M_{12}) \end{aligned}\right\} \ldots\ldots\ldots\ldots (25)$$

in which,

$$\left.\begin{aligned} c &= 1 \mp \frac{(Q\,L)^2}{15} + 2\,(Q\,L)^4 \mp \frac{(Q\,L)^6}{1\,575} + \cdots \\ &\text{and,} \\ d &= 1 \mp \frac{7\,(Q\,L)^2}{60} + \frac{3\,(Q\,L)^4}{2\,520} \mp \frac{127\,(Q\,L)^6}{100\,800} + \cdots \end{aligned}\right\} \ldots\ldots (26)$$

the upper signs in Equation (26) being used when P is tension, and the lower when P is compression.

Relation Between the Deflection Angles of Members Meeting in a Joint.— Let any number of members, $n\,1$, $n\,2 \ldots nm$, meet at a joint, n. It may readily be shown* that the τ of any of these members can be expressed in terms of the τ of any other, chosen for purpose of reference, and the angular changes taking place between the different members at n.

In general,

$$\tau_{nm} = \tau_{n1} + \sum_{1}^{m-1} \Delta\,\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

It is usually convenient to take as the reference, τ, that of the member first reached in passing around the joint from the outside of the frame in a clockwise direction.

The Sum of the Bending Moments About Any Joint is Zero.—In order to find these unknown τ's, use is made of the fact that each joint is in equilibrium under all the bending moments about the joint. If an eccentric external force were present, it can be accounted for, too, but for the time being it is assumed that there is none.

Hence, from Fig. 6:

$$M_{n_1} + M_{n_2} + \ldots + M_{nm} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

in which, M_{n_1} is the moment at n in Member $n1$, etc.

Referring now to Equation (23):

$$M_{n1} = \frac{2\,E\,I_{n1}}{L_{n1}}\,(2\,a_{n1}\,\tau_{n1} + b_{n1}\,\tau_{1n})$$

in which,

I_{n_1} = the moment of inertia of Member $n1$;
L_{n_1} = the length of Member $n1$;
τ_{n_1} = the deflection angle of Member $n1$ at End n;
τ_{1n} = the deflection angle of Member $n1$ at End 1.

$$a_{n1} = 1 \pm \frac{(Q_{n1}\,L_{n1})^2}{30} - \frac{11\,(Q_{n1}\,L_{n1})^4}{25\,000} \pm \ldots$$

$$b_{n1} = 1 \mp \frac{(Q_{n1}\,L_{n1})^2}{60} + \frac{13\,(Q_{n1}\,L_{n1})^4}{25\,000} \mp \ldots$$

the upper signs being used when P is tension, and the lower when P is compression, a and b being constants.

$$Q_{n1}{}^2 = \frac{P_{n1}}{E\,I_{n1}}$$

in which, P_{n_1} is the total primary stress transmitted by Member $n1$.

Similarly,

$$M_{n2} = \frac{2\,E\,I_{n2}}{L_{n2}}\,(2\,a_{n2}\,\tau_{n2} + b_{n2}\,\tau_{2n})$$

in which the various quantities have exactly the same significance as before, and so on for the other bending moments.

* "Modern Framed Structures," by Messrs. Johnson, Bryan and Turneaure, Part II.

There is, therefore, for every joint an equation that will contain the deflection angles of all the members that enter the joint. Furthermore, this equation is linear. For example, Equation (28) will take the form:

$$\left.\begin{array}{l} c_{n1}\,\tau_{n1} + c_{n2}\,\tau_{n2} + c_{n3}\,\tau_{n3} + c_{n4}\,\tau_{n4} + \ldots \\ + c_{1n}\,\tau_{1n} + c_{2n}\,\tau_{2n} + c_{3n}\,\tau_{3n} + c_{4n}\,\tau_{4n} + \ldots = 0 \end{array}\right\} \ldots\ldots\ldots(29)$$

in which, c_{n1}, etc., are constants.

Referring now to Equation (27), the τ's of Equation (29) must be changed into what are termed the "reference deflection angles" of the joints, of which Equation (29) makes use. For the purpose of consistency, take each member separately in a clockwise direction around every joint beginning outside the truss; the τ of the member first met will be the reference deflection angle of the joint. Taking τ_{n_1} of Equation (29) as the reference deflection angle of the joint, n (Fig. 6), call it τ_n. Making use of Equation (27),

$$\tau_{n2} = \tau_n + \Delta\,\alpha_{1n_2}$$

in which, α_{1n_2} is the angle between $1n$ and $2n$,

$$\tau_{n3} = \tau_n + \Delta\,\alpha_{1n_2} + \Delta\alpha_{2n_3}, \text{ etc.},$$

while,

$$\tau_{1n} = \tau_1 + \Sigma\,\Delta\,\alpha_1,$$

in which, τ_1 is the reference deflection angle for Joint (1), and $\Sigma\,\Delta\,\alpha_1$ is the sum of the changes of angles that lie between the reference member of Joint (1) and the member, $n1$, etc.

Summarizing, it is possible to change Equation (29) into an equation, as follows:

$$k_n\,\tau_n + k_1\,\tau_1 + k_2\,\tau_2 + k_3\,\tau_3 + k_4\,\tau_4 + \ldots\; S = 0 \ldots\ldots\ldots(30)$$

where the k's and S's are all constants.

A repetition of this for every joint will finally give a set of simultaneous linear equations, equal in number to the joints, and the unknowns are the reference deflection angles. On solving these, substitute the values in Equation (27) to determine all the deflection angles in the truss. Making use of the latter in Equation (23), the two bending moments at the ends of every member are found. A check must be applied here; the sum of the bending moments at every joint must be zero. If this is satisfactorily done, the secondary stresses are obtained from the formula,

$$f = \frac{M\,y}{I},$$

in which,

f = the secondary stress;
M = the bending moment at one end of a particular member;
y = the distance of the extreme fiber from the center of gravity; and
I = the moment of inertia of the member in question.

It must be remembered that the extreme fiber represents the set of extreme fibers that lie in a plane parallel to the neutral plane of the member.

Mathematically, the problem is solved. In practice, however, the labor is excessive; certain approximations, therefore, are necessary.

2.—Modification of Manderla's Method

An improvement in Manderla's method is due to Winkler. The analysis is much the same as that just explained with no change as far as the derivation of Equation (3).

The Bending Moments Introduced at the Joints.—According to Equation (3) and Fig. 5:

$$M_x = M_1 - Py - V_1 x_1$$

Now, y, the offset of N from the line, A-B, is generally so small that $-Py$ may be neglected. Therefore,

$$M_x = M_1 - V_1 x \qquad (31)$$

giving rise to the following equation of the elastic line:

$$E\,I \, . \, \frac{d^2 y}{d x^2} = V_1 x - M_1 \qquad (32)$$

Equation (32) can be easily solved by direct integration. Integrating once with respect to x,

$$E\,I \frac{d\,y}{d\,x} = + \frac{V_1 x^2}{2} - M_1 x + C \qquad (33)$$

which represents the slope of the elastic line at any point, N, distant x from A. The constant, C, will be determined later. Integrating again,

$$E\,I \, . \, y = + \frac{V_1 x^3}{6} - \frac{M_1 x^2}{2} + C x + D \qquad (34$$

which gives the distance of the point, N, from AB. The constants, C and D, are now determined from the fact that $y = 0$ when $x = 0$ and $x = L$. Therefore,

$$D = 0, \text{ and } C = + \frac{M_1 L}{2} - \frac{V_1 L^2}{6}$$

Substituting now in Equation (33), the equation of the slope of the elastic line becomes:

$$E\,I \, . \, \frac{d\,y}{d\,x} = + \frac{V_1 x^2}{2} - M_1 x + \frac{M_1 L}{2} - \frac{V_1 L^2}{6} \qquad (35)$$

The slope at $x = 0$ is the deflection angle, τ_1. Hence,

$$E\,I \, . \, \tau_1 = + \frac{M_1 L}{2} - \frac{V_1 L^2}{6} \qquad (36)$$

The moments about B are now taken, giving $V_1 \; L = M_1 + M_2$. Hence, Equation (36) becomes:

$$\tau_1 = \frac{L}{6\,E\,I} (2\,M_1 - M_2) \qquad (37)$$

Starting at the other end of the member, there would have resulted a similar expression,

$$\tau_2 = \frac{L}{6\,E\,I} (2\,M_2 - M_1) \qquad (38)$$

(Compare Equations (37) and (38) with Equations (25); the difference is that the c and d of Equations (25) are each replaced by unity in Equations (37) and (38). In other words, the Q in Equations (26) is here considered as zero, that is, the force, P, is regarded as zero. That is exactly what would be expected, for at the start of this analysis the effect of the direct stress on the bending moments was disregarded.)

Making use of Equations (37) and (38) to express M_1 and M_2 in terms of τ_1 and τ_2:

$$\left.\begin{aligned} M_1 &= \frac{2\,E\,I}{L}\,(2\,\tau_1 + \tau_2) \\ M_2 &= \frac{2\,E\,I}{L}\,(2\,\tau_2 + \tau_1) \end{aligned}\right\} \dots\dots\dots\dots\dots (39)$$

(Again, comparing with Equations (23), it is evident that the a and b are each replaced by unity, for the same reasons that c and d were.)

Relation Between the Deflection Angles of Members Meeting in a Joint.— This is the same as was given for Manderla's original method.

The Sum of the Bending Moments Around Any Joint is Zero.—Again, as previously (see Fig. 6):

$$M_{n_1} + M_{n_2} + M_{n_3} + M_{n_4} = 0$$

Making use of Equations (39), this becomes:

$$\left.\begin{aligned} &\frac{2\,E\,I_{n1}}{L_{n1}}\,(2\,\tau_{n1} + \tau_{1n}) + \frac{2\,E\,I_{n2}}{L_{n2}}\,(2\,\tau_{n2} + \tau_{2n}) \\ &\quad + \frac{2\,E\,I_{n3}}{L_{n3}}\,(2\,\tau_{n3} + \tau_{3n}) + \frac{2\,E\,I_{n4}}{L_{n4}}\,(2\,\tau_{n4} + \tau_{4n}) = 0 \end{aligned}\right\} \dots\dots (40)$$

Let the quantity, $\dfrac{I}{L}$, be called K. Hence,

$$K_{n1}\,(2\,\tau_{n1} + \tau_{1n}) + K_{n2}\,(2\,\tau_{n2} + \tau_{2n}) + K_{n3}\,(2\,\tau_{n3} + \tau_{3n}) + K_{n4}\,(2\,\tau_{n4} + \tau_{4n}) = 0 \dots\dots\dots\dots\dots\dots (41)$$

Use may then be made of the reference deflection angles. Thus, in Fig. 6, the τ of the joint, n, is τ_{n_1}, which without fear of confusion may be called τ_n, while τ_{n_2} becomes $\tau_n + \Delta\,\alpha$, τ_{n_3} becomes $\tau_n + \Delta\,\alpha + \Delta\,\beta$, etc. (See Equation (27)).

As many equations of the type of Equation (41) are found as there are joints. There will be a large number of τ's of the nature, $\tau_{n_1}, \tau_{n_2} \dots \tau_{1n}, \tau_{2n} \dots$, etc., but by making use of Equation (27), there results finally a set of as many simultaneous linear equations in the reference τ's as there are joints. After solving these, Equation (39) will express the bending moments at the ends of all the members. Finally, the use of the equation, $f = \dfrac{M\,y}{I}$, gives the required secondary stresses.

Professor Turneaure's method will be followed in arriving at the set of simultaneous linear equations for the τ's, and the steps that lead to the equations with comparative ease will be explained more fully in the practical application which Manderla's original method lacked.

It is evident that the value of E does not affect the secondary stresses, for the simple reason that in this analysis E appears in the denominator of an expression in one place, only to be cancelled out in the numerator later. As all the methods that will be applied show this property, E will be considered as unity throughout. It is another matter, of course, in testing the theory in the field; there, in determining what stress a certain deformation represents, E is essential.

Manderla solves his simultaneous equations by a method of approximation. It will be found that the coefficient of one of the τ's in every equation (the one belonging to the reference, τ, of the joint for which that particular equation is formed) is much larger than any of the others. By first imagining all the small coefficients to be zero, approximate values are obtained for the τ's that are retained. Taking each equation as given, and substituting in all the terms except that with the large coefficient the approximate values of the τ's just found, a fresh set of values for the τ's is determined. This process is repeated until a stage is reached where the τ's of the final approximation differ little from those of the previous approximation—these are then the values required. This method lends itself to slide-rule computation and is, therefore, much in favor with engineers; the great objection is that many approximations are necessary when there are many joints—a tedious task. This method, however, will be illustrated by an example.

Professor Turneaure has a definite method of elimination (25). Winkler has also given such a method (4). Several investigators have noted the fact, that all these sets of simultaneous equations are, as they appear, of the type known as Gauss' normal equations. Most books on least squares describe Gauss' method of solving a set of simultaneous linear equations, in which he re-arranges them in what he calls the "normal" form, to which he then applies a most effective and absolutely self-checking method of solution. A solution of secondary stresses, using Gauss' method of normal equations, will also be given.

3.—Müller-Breslau's Method

Introduction.—A simple truss made up of triangular elements is such that, if it has n joints, the number of members is $(2n\text{-}3)$. This is the criterion of what is known as a statically determinate plane framework. In the present problem every member has a certain bending moment at each end; these bending moments are really the unknown quantities that are to be determined. Their number is, therefore, $2\ (2n\text{-}3) = (4n\text{-}6)$.

The underlying principles of this method are exactly those of the modified Manderla method; the manner of attacking the problem is different. As many linear equations as there are joints are obtained from the property that the sum of the moments about a joint is zero. As will be shown later, there are three independent linear relations between the three changes of angle in every triangular element and the six moments, two for each of the three sides. As there are $(n\text{-}2)$ triangular elements, the total number of these relations is $3\ (n\text{-}2) = (3n\text{-}6)$. There are, therefore, $n + (3n\text{-}6) = (4n\text{-}6)$ linear equations to solve the $(4n\text{-}6)$ unknown bending moments.

Manderla has this number of equations too, but, by making use of the fact that all the deflection angles at a joint can be expressed in terms of one at that joint, he is able to tie up, as it were, the ($3n$-6) relations in the n simultaneous equations; after these are solved he reverts to the expression for all the τ's at a joint in terms of the one, and then uses the remaining ($3n$-6) relations.

Müller-Breslau has a special way of setting down the total ($4n$-6) relations, one after the other. By assuming two particular bending moments at the left end of a truss to be known, he proceeds to express every successive one in terms of the preceding ones until, finally, he arrives at two equations in which every bending moment is already in terms of the two assumed ones. In other words, he obtains two linear equations with which to find the bending moments. Repeating the work and substituting these in the expressions, all the bending moments become known. It is a neat method of elimination.

All moments in a clockwise direction will be reckoned positive in this method.

Procedure.—A triangle, ABC (Fig. 7) under such a system of moments will be bent in a form as shown. Hence,

$$\tau_2 + \alpha + \Delta\,\alpha = \tau_3 + \alpha.$$

Therefore,

$$\Delta\,\alpha = \tau_3 - \tau_2.$$

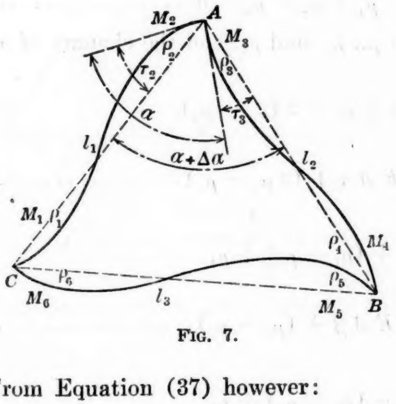


FIG. 7.

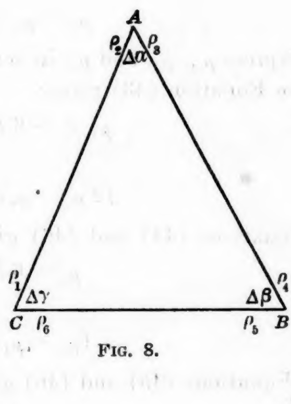


FIG. 8.

From Equation (37) however:

$$\tau_2 = 2\,M_2 - \frac{M_1}{6\,E\,I_1}\,.\,l_1$$

and,

$$\tau_3 = 2\,M_3 - \frac{M_4}{6\,E\,I_2}\,.\,l_2$$

(The analysis for the deflection angles in terms of the end bending moments is of course the same as that used in the modified Manderla method. Note that Müller-Breslau also neglects the effect of the primary stress on the bending moment.)

Therefore,

$$\Delta\,\alpha = 2\,M_3 - \frac{M_4}{6\,E\,I_2}\,.\,l_2 - 2\,M_2 - \frac{M_1}{6\,E\,I_1}\,.\,l_1 \text{..........} (42)$$

For convenience let,

$$\frac{M_1\,l_1}{I_1} = \rho_1;\ \frac{M_2\,l_1}{I_1} = \rho_2;\ \frac{M_3\,l_2}{I_2} = \rho_3;$$

$$\frac{M_4\,l_2}{I_2} = \rho_4;\ \frac{M_5\,l_3}{I_3} = \rho_5;\ \frac{M_6\,l_3}{I_3} = \rho_6$$

Insert these expressions in Equation (42) and multiply by $6\,E$, obtaining:

$$6\,E.\,\Delta\,\alpha = (2\,\rho_3 - \rho_4) - (2\,\rho_2 - \rho_1),$$

or,

$$6\,E.\,\Delta\,\alpha = \rho_1 - 2\,(\rho_2 - \rho_3) - \rho_4 \text{..................} (43)$$

Similarly,

$$6\,E.\,\Delta\,\beta = \rho_3 - 2\,(\rho_4 - \rho_5) - \rho_6 \text{..................} (44)$$

and,

$$6\,E\,.\,\Delta\,\gamma = \rho_5 - 2\,(\rho_6 - \rho_1) - \rho_2 \text{..................} (45)$$

Between the changes of the angle, $\Delta\,\alpha$, $\Delta\,\beta$, and $\Delta\,\gamma$, of the triangle there exists the relation:

$$\Delta\,\alpha + \Delta\,\beta + \Delta\,\gamma = 0$$

or,

$$\rho_1 - \rho_2 + \rho_3 - \rho_4 + \rho_5 - \rho_6 = 0 \text{................} (46)$$

Express ρ_4, ρ_5, and ρ_6, in terms of ρ_1, ρ_2, and ρ_3, and the changes of angle. Hence Equation (43) gives:

$$\rho_4 = -\,6\,E\,.\,\Delta\,\alpha + \rho_1 - 2\,(\rho_2 - \rho_3)$$

or,

$$(2\,\rho_3 - \rho_4) = 6\,E\,\Delta\,\alpha + (2\,\rho_2 - \rho_1) \text{...............} (47)$$

Equations (44) and (46) give:

$$\rho_5 = 6\,E\,\Delta\,\beta + (\rho_1 - \rho_2) + \rho_4$$

or,

$$(\rho_5 - \rho_4) = 6\,E\,\Delta\,\beta + (\rho_1 - \rho_2) \text{...............} (48)$$

and Equations (45) and (46) give:

$$\rho_6 = 6\,E\,\Delta\,\gamma - (\rho_3 - \rho_4) + \rho_1$$

or,

$$(\rho_1 - \rho_6) = 6\,E\,\Delta\,\gamma + (\rho_3 - \rho_4) \text{...............} (49)$$

In using Equations (47), (48), and (49), it is convenient to bear in mind the following rules:

1.—When any three adjacent values, ρ_1, ρ_2, and ρ_3, are known, the value, ρ_4, is readily obtained by Equation (47), in which $(2\,\rho_3 - \rho_4)$ is the deflection angle of *AB* at *A*, that is, of the member first met in passing around the joint, *A*, in a clockwise direction, and $(2\,\rho_2 - \rho_1)$ is the deflection angle of *AC* at *A*.

2.—When any four adjacent values, ρ_1, ρ_2, ρ_3, and ρ_4, are given, a fifth value, ρ_5 or ρ_6, may be found from Equation (48) or Equation (49), in which $(\rho_5 - \rho_4)$ is the difference of the ρ-values at B of the sides, BC and BA, which include the angle, B, and $(\rho_1 - \rho_2)$ is the difference of the values of the side, AC, opposite to B.

In both cases the difference is found by passing around the joint in a clockwise direction and then subtracting the second one of the values in question from the first.

Equations (47), (48), and (49) and the three fundamental equations, one for each joint, namely, that the sum of the bending moments around each joint is zero, comprise six independent relations between the six unknowns, $\rho_1 \ldots \rho_6$, which therefore can be determined.

(This is for the case of a structure composed of a single triangle. Note that the changes of angle are previously determined in exactly the same manner as was shown for Manderla's method.)

If instead of a single triangle the structure is as shown in Fig. 9, the process of calculating the values, $\rho_1 \ldots \rho_{14}$, is as follows: Consider any Joint (1) in which only two members intersect, and assume for the present that the ρ_1 and ρ_2 of the member, 1-2, are known.

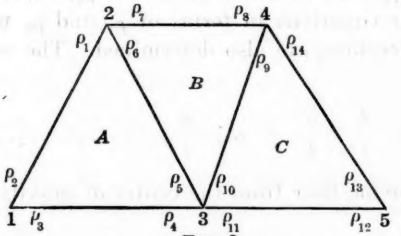


FIG. 9.

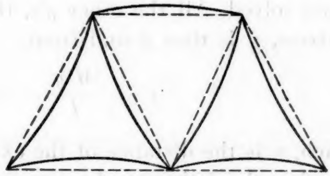


FIG. 10.

The fundamental equation, $M = 0$, for Joint (1) gives:

$$M = M_2 + M_3 = 0$$

Turning back to the notation adopted, this equation becomes

$$\rho_3 \cdot \frac{I_2}{l_2} = -\rho_2 \cdot \frac{I_1}{l_1}$$

After ρ_3 is known, its value is substituted in Equation (47), thus giving ρ_4 as a function of ρ_1 and ρ_2. By substituting the values of ρ_3 and ρ_4 in Equations (48) and (49), ρ_5 and ρ_6 also are expressed in terms of ρ_1 and ρ_2.

In passing to Joint (2), ρ_7 is the only quantity not known; but,

$$M = M_7 + M_6 + M_1 = 0$$

or,

$$\rho_7 \cdot \frac{I_4}{l_4} + \rho_6 \cdot \frac{I_3}{l_3} + \rho_1 \cdot \frac{I_1}{l_1} = 0$$

whence, ρ_7 is expressed in terms of ρ_1 and ρ_2.

By applying Equations (47), (48), and (49) to the triangle, *B*, of which ρ_5, ρ_6, and ρ_7 are now expressed in terms of ρ_1 and ρ_2, the remaining quantities, ρ_8, ρ_9, and ρ_{10}, may first be expressed in terms of ρ_5, ρ_6, and ρ_7, and then, in turn, in terms of ρ_1 and ρ_2.

Knowing ρ_4, ρ_5, and ρ_{10}, ρ_{11} at Joint (3) is found from,

$$M_4 + M_5 + M_{10} + M_{11} = 0$$

and, finally, ρ_{12}, ρ_{13}, and ρ_{14}, are found by means of Equations (47), (48), and (49), which are to be applied to the triangle, *C*.

The preceding analysis shows that each ρ may be expressed as a linear function of ρ_1 and ρ_2, which were assumed to be known at the start. There are two equations left that have not been used, namely,

$$\Sigma\, M_{(4)} = \Sigma\, M_{(5)} = 0$$

These are:

$$\left.\begin{aligned} &\rho_{14} \cdot \frac{I_7}{l_7} + \rho_9 \cdot \frac{I_5}{l_5} + \rho_8 \cdot \frac{I_4}{l_4} = 0 \\ \text{and,}\quad &\\ &\rho_{12} \cdot \frac{I_6}{l_6} + \rho_{13} \cdot \frac{I_7}{l_7} = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (50)$$

Making use of the expressions for ρ_{14}, etc., in terms of ρ_1 and ρ_2, Equations (50) become two simultaneous linear equations in terms of ρ_1 and ρ_2 which are then solved. All the other ρ's, therefore, are also determined. The secondary stress, *f*, is then found from,

$$f = \frac{M\, y}{I} = \rho \cdot \frac{I}{l} \cdot \frac{y}{I} = \rho \cdot \frac{y}{l} \ldots\ldots\ldots\ldots\ldots (51)$$

in which, y is the distance of the extreme fiber from the center of gravity and l is the length of the member.

Remarks.—Müller-Breslau's method is not exactly like this one, which is due to Páez. Müller-Breslau assumes that the triangles alternate in shape as in Fig. 10.

This requires a changing of signs in the moments passing from one triangle to the next, a method that is apt to create confusion and is the source of many errors. The method of Páez is to reckon all the moments to be positive and to remain positive in the analysis. It is then known what shapes the triangles assume theoretically. If, in the calculations, a moment at one end of a member is found to be negative the shape of the member should be contrary to that assumed. Thus, Páez' modification is far superior in that the signs are taken care of in the work.

4.—Ritter's Method.

Introduction.—The graphical method of computing secondary stresses developed by Ritter is also based on the fundamental equation of Manderla. Its aim is to solve graphically the (4*n*-6) equations for the (4*n*-6) bending moments, one at each end of every member.

Procedure.—Consider any joint of a structure as Joint (5) in Fig. 11, at which the four members, 5-3, 5-4, 5-6, and 5-7, meet. Imagine all the members to be bent under positive (clockwise) bending moments. The two bending moments of the member, 3-5, are designated M_3 at Joint (5) and M_3' at Joint (3). For the member, 4-5, the moments are M_4 at Joint (5) and M_4' at Joint (4), etc. When Joint (4) is considered, the bending moments of the member, 4-5, are M_5 at Joint (5) and M_5' at Joint (5).

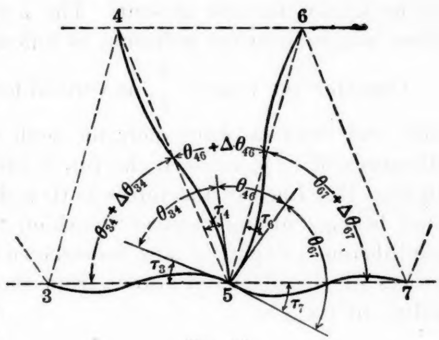


FIG. 11.

It follows, therefore, that:

$$M_4 \text{ for Joint } (5) = M_5' \text{ for Joint } (4)$$
$$M_4' \text{ “ “ } (5) = M_5 \text{ “ “ } (4)$$

which means that each moment in a truss is designated in two different ways.

From Fig. 11,

$$\theta_{3.4} + \Delta\,\theta_{3.4} + \tau_4 = \theta_{3.4} + \tau_3$$

or,

$$\Delta\,\theta_{3\cdot 4} = \tau_3 - \tau_4$$

Substituting the values of τ_3 and τ_4 as found from Equation (37),

$$E\,.\,\Delta\,\theta_{3.4} = \frac{2\,M_3 - M_3'}{6\,I_3}\,.\,l_3 - \frac{2\,M_4 - M_4'}{6\,I_4}\,.\,l_4$$

in which, M_3 and M_4 are the moments at the near ends, and M_3' and M_4' those at the far ends of the members, 5-3 and 5-4, which include the angle, $\theta_{3\cdot 4}$. Similar expressions can be written for $E\,.\,\Delta\,\theta_{4\cdot 6}$ and $E\,.\,\Delta\,\theta_{6\cdot 7}$.

The sum of the moments around Point (5) is zero and, therefore,

$$M_3 + M_4 + M_6 + M_7 = 0.$$

For convenience, let $\frac{M\,l}{6\,I} = \mu$. The four equations connected with Joint (5) can now be written as follows:

$$\left.\begin{aligned} E\,.\,\Delta\,\theta_{3.4} &= (2\,\mu_3 - \mu_3') - (2\,\mu_4 - \mu_4') \\ E\,.\,\Delta\,\theta_{4.6} &= (2\,\mu_4 - \mu_4') - (2\,\mu_6 - \mu_6') \\ E\,.\,\Delta\,\theta_{6.7} &= (2\,\mu_6 - \mu_6') - (2\,\mu_7 - \mu_7') \\ \frac{I_3}{l_3}\,.\,\mu_3 + \frac{I_4}{l_4}\,.\,\mu_4 + \frac{I_6}{l_6}\,.\,\mu_6 &+ \frac{I_7}{l_7}\,.\,\mu_7 = 0 \end{aligned}\right\} \qquad (52)$$

For every joint there are as many equations as there are members intersecting at the joint. The total number of these equations is equal to the number of ends of all the members and, therefore, equal to that of the unknown μ-values.

The changes of angle are determined either graphically or analytically by Ritter's method as used previously,* and are, therefore, regarded as known.

---

* See p. 977.

In order to explain how Ritter solves his problem, assume all the μ'-values to be known for the present. The μ-values are then determined easily by force and equilibrium polygons, as follows:

Consider the values, $\frac{I}{l}$, as vertical forces; plot them in order on a vertical line and draw a force polygon with any arbitrary pole, Fig. 12. The distances, $E \,.\, \Delta\,\theta$, between the forces are then laid out horizontally as shown in Fig. 13. Each of the forces is then displaced to the left by the amount, μ', that belongs to the member to which the so-called force corresponds. The equilibrium polygon is now constructed for these displaced forces. The distances of the displaced forces from their resultant will then be the double values of the μ's.

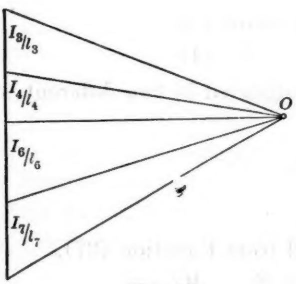


FIG. 12.

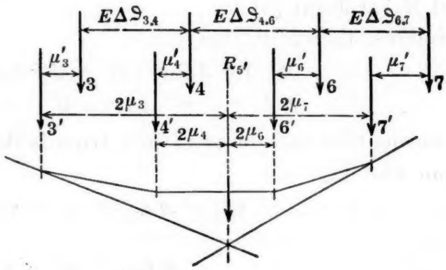


FIG. 13.

A study of Equations (52) and Fig. 13 will prove the truth of this statement. The distance of a displaced force from the resultant, is $2\,\mu$ for a displacement, μ'; consequently, the distance of an undisplaced force from the resultant of the displaced forces is equal to $(2\,\mu - \mu')$, and the difference between the distances of two adjacent undisplaced forces must equal $E \,.\, \Delta\,\theta$. The first three of the Equations (52) are, therefore, satisfied. The fourth is also satisfied, as the algebraic sum of the moments $(= \Sigma \frac{I}{l} \,.\, 2\,\mu)$ of the parallel forces, $\frac{I}{l}$, about any point on the line of the resultant is equal to zero.

The difficulty, however, is that neither the μ-values nor the μ'-values are known. Let Figs. 14 and 15 represent the equilibrium polygons for Joints (4) and (5), R_4' and R_5', the resultants of the displaced forces, 2', 3' ..... If, now, the distance between an undisplaced force and the resultant of the displaced forces is denoted by γ, we have:

$$2\,\mu_4 = \mu_4' + \gamma_4$$
$$2\,\mu_5 = \mu_5' + \gamma_5$$

but,

$$\mu_4 = \mu_5' \text{ and } \mu_5 = \mu_4'$$

therefore,

$$2\,\mu_5' = \mu_4' + \gamma_4$$
$$2\,\mu_4' = \mu_5' + \gamma_5$$

By solving these equations:

$$\left.\begin{aligned} \mu_4' &= \frac{(\gamma_4 + 2\,\gamma_5)}{3} \\ \text{and,}\qquad \\ \mu_5' &= \frac{(\gamma_5 + 2\,\gamma_4)}{3} \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(53)$$

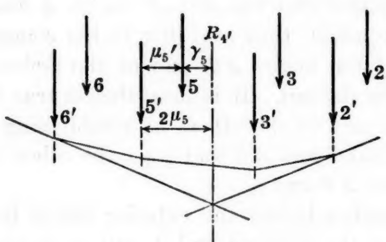


FIG. 14.

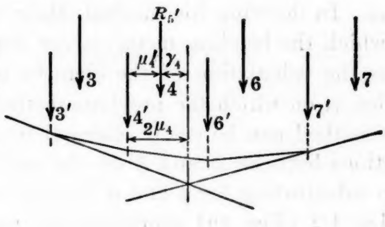


FIG. 15.

If the positions of the resultants, R_4' and R_5', of the displaced forces were known, the values of $2\,\mu$ could be found at once. These positions can only be found by trial. As the position of the resultant, R', undergoes little displacement (that is, the changes of the values, γ, are generally small), a roughly approximate position of the resultant, R', may be assumed as that of the resultant, R, of the undisplaced forces.

Measuring now the distances of the undisplaced forces from their resultant, and calling these distances the first approximation to the true γ's, these values are substituted in Equations (53) to give the first approximation to the μ's, the amounts by which the forces are displaced. Next determine the resultant of this first set of displaced forces.

The distances of the undisplaced forces to the resultant of the first set of displaced forces are the second approximation to the γ-values. Carrying through the process, a second resultant of displaced forces is found. This process is repeated several times, until it is found that, for every joint, the resultant of the displaced forces from the last trial differs very little from that of the former trial. The values of $2\,\mu$ are then the distances of the last set of displaced forces from their resultant.

The secondary stress is, therefore,

$$f = \frac{M\,y}{I} = \mu\,.\,\frac{6\,I}{l}\,.\,\frac{y}{I} = \frac{6\,y}{l}\,.\,\mu\ldots\ldots\ldots\ldots\ldots\ldots(54)$$

Remarks.—In order to avoid errors strict attention must be paid to the signs of $\Delta\,\theta$, μ, and μ'. The succession of the members around a joint should be taken in a clockwise direction; a $\Delta\,\theta$ which has a positive sign should be laid out at the right, and a negative $\Delta\,\theta$ to the left. Further, any quantity, μ or μ', to be transferred into a corresponding equilibrium polygon should be

laid out to the left, if it is situated to the left of the resultant of the forces, $\frac{I}{l}$, and *vice versa*. Finally the values, $2\ \mu$, situated to the left of the resultant are positive and those to the right are negative.

5.—Mohr's Semi-Graphical Method

Introduction.—The preceding methods of Manderla, Müller-Breslau, and Ritter are based on the equations giving the changes of angle in the triangular elements, and on the equation arising from the fact that the sum of the bending moments around a joint is zero; these were the two sets of fundamental equations. In deriving his method, Mohr introduced two sets of angles, ψ and ϕ, on which the bending moments are dependent, thus avoiding in his computations the calculation of the changes of the angle, $\Delta\theta$, and of the deflection angles, τ, on which the previous methods depend. It is nevertheless true that this method can be easily changed into any of the others by establishing the relations between ψ and ϕ on the one hand and $\Delta\theta$ and τ on the other, and then substituting for ψ and ϕ in terms of $\Delta\theta$ and τ.

Let 1-2 (Fig. 16) represent any member before the exterior forces begin to act. Under the action of any loading the joints, 1 and 2, will be displaced to 1′ and 2′. The lines, 1′-2″ and 1″-2′, are parallel to each other and to the member, 1-2; 1′ T_1 and 2′ T_2 indicate the end tangents drawn to the elastic line of the curved member after deformation has taken place.

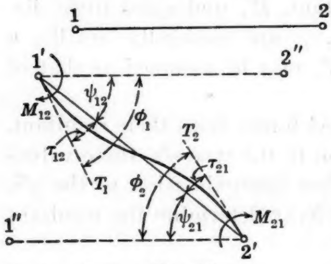


Fig. 16.

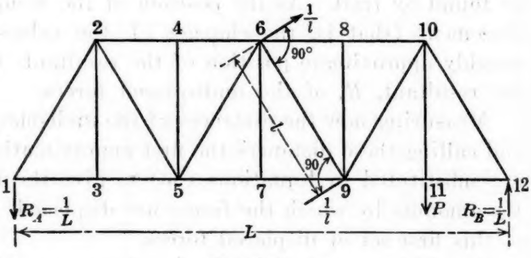


Fig. 17.

The angle, $2'' - 1' - 2' = \psi_{12} = \psi_{21}$, included between the original direction of the member and the chord, 1′-2′, of the curved member after deformation, is the angle through which a member turns under load, assuming that the truss is provided with frictionless joints. More clearly expressed, suppose the truss is one with frictionless joints; the load causes it to deflect, taking a member like 1-2 into a different position, namely, 1′-2′, the member remaining straight. Now, suppose that the gusset-plates at the ends, 1′ and 2′, start to act, introducing the bending moments, M_{12} and M_{21}, and causing the member to become curved. The assumption is that this bending takes place in such a manner, that the ends, 1 and 2, of the member are not moved. Mohr calls the angle, ψ_{12} and ψ_{21}, "the change in slope".

The angle, ϕ_1, included between the original direction, 1-2 or 1′-2″, of the member and its end tangent, 1′ T_1, is called by Mohr "the angle of rotation of the joint". This angle is the same for the ends of all members at

any joint, owing to the assumption that the ends are riveted rigidly together. In other words, the angle between any two members meeting at a joint is assumed to be constant during deformation of the truss.

To express the bending moments as functions of ϕ and ψ, start with the fundamental Equation (39),

$$M_{12} = \frac{2\,E\,I}{l} \cdot (2\,\tau_{12} + \tau_{21}).$$

From Fig. 16,

$$\tau_{12} = \phi_1 - \psi_{12},$$

and,

$$\tau_{21} = \phi_2 - \psi_{21} = \phi_2 - \psi_{12}.$$

Substituting these values in the fundamental equation,

$$\left.\begin{aligned} M_{12} &= \frac{2\,E\,I}{l} \cdot (2\,\phi_1 - 2\,\psi_{12} + \phi_2 - \psi_{12}) \\ \text{that is,}\quad M_{12} &= \frac{2\,E\,I}{l} \cdot (2\,\phi_1 + \phi_2 - 3\,\psi_{12}) \\ \text{and, similarly,}\quad M_{21} &= \frac{2\,E\,I}{l} \cdot (\phi_1 + 2\,\phi_2 - 3\,\psi_{12}) \end{aligned}\right\} \ldots\ldots\ldots\ldots(55)$$

The quantities, ψ, are really the properties of the truss considered as having frictionless joints, and are easily obtained as the truss is statically determinate. It has been proved that there is only one ϕ for each joint; also, the sum of the moments around a joint is zero. Hence, if this fact is expressed for each joint, making use of Equations (55), there will result as many simultaneous linear equations involving ϕ's as there are joints, that is, as there are ϕ's, the ψ's having been determined previously. These equations are solved for the ϕ's. The secondary stresses then follow easily.

Calculation of the Changes in the Slope, ψ.—This calculation may be done analytically or graphically.

(1).—Analytical Method.—The values of ψ are obtained analytically from the equation, $M\,\psi = \Sigma\, s\,.\,\Delta\, l$, or, by assuming $M = 1$,

$$1\,.\,\psi = \Sigma\, s\,.\,\Delta\, l \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(56)$$

in which, $M = 1$ designates any moment acting on a truss, s the stress in a member due to the moment, and ψ the angle of rotation of the member under the loads that produce the change in length, $\Delta\, l$, of the member in question.

Equation (56) follows from the theorem that, if a system in equilibrium is acted on by a set of external forces, thereby causing the system to move into another position of equilibrium, the work done by the external forces is equal to that done by the internal stresses, these stresses having been called into existence by the externally applied forces.

To illustrate the analytical method of determining s, consider the truss shown in Fig. 17. Suppose the change in the slope of the member, 6-9, under the action of the loading, P, at Joint (11), is required. For this purpose, first assume that P does not exist, but that two forces, $\frac{1}{l}$, act at the ends

of the member, 6-9, as shown (Fig. 17), introducing a couple of moment $= +1$. This moment requires a downward left-hand reaction, $\frac{1}{L}$, and an upward right-hand reaction of the same magnitude.

Compute the stresses, s, set up in the members under these two pairs of forces, namely, $\frac{1}{l}$ and $\frac{1}{L}$; also compute all the Δl-values of the members caused by the given loading, P, at Joint (11). The products, $s \,.\, \Delta\, l$, are then computed for each member and added, giving the value, $\Sigma\ s\ .\ \Delta\ l$, which is then the angle of revolution or change in the slope, ψ, of the member, 6-9, due to the load, P, at Joint (11).

It must be evident that the calculation of all the slopes of all the members—a process of repeating the method given as many times as there are members—is extremely laborious. It is safe to say that engineers will always avoid the analytical method and make use of the graphical, when the results are obtained with greater ease and reasonable accuracy.

(2).—Graphical Method.—The graphical method of determining the ψ-values consists in the construction of a Williot displacement diagram for the truss under the given loading, that is, the truss with the imaginary frictionless joints.

For the sake of simplicity consider any triangle, ABC, Fig. 18, and imagine that some loading produces the stresses, $-s_1$, $+s_2$, and $+s_3$, in BC, CA, and AB, thus changing their lengths, l_1, l_2, and l_3, by the amounts, $-\Delta\, l_1$, $+\Delta\, l_2$, and $+\Delta\, l_3$.

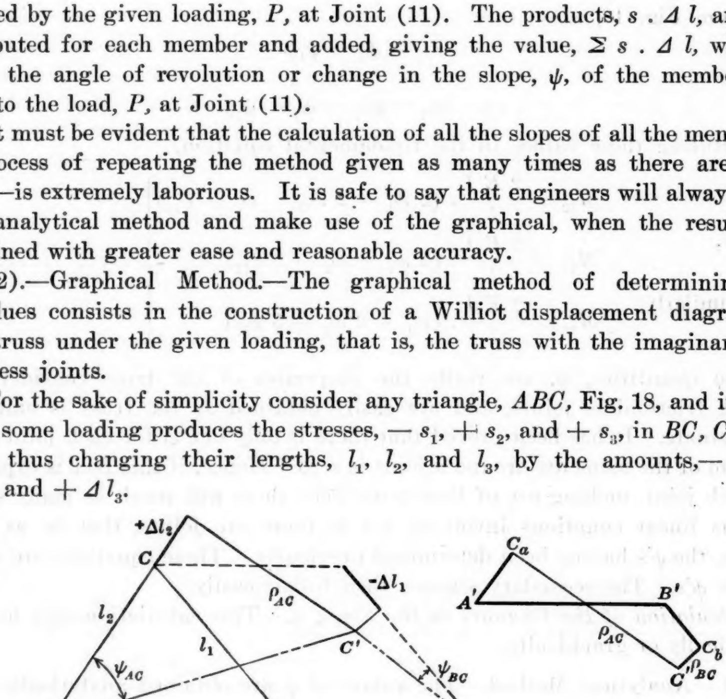


FIG. 18.

By regarding the joint, A, and the direction, A to B, as fixed, the displaced positions, B' and C', of B and C are found as shown. From Fig. 18, and also because the elastic deformations, $\Delta\, l$, of the members are small, so that the tangent of the arc may be substituted for the arc itself,

$$\rho_{AC} = l_2\, \psi_{AC}$$

or,

$$\psi_{AC} = \frac{\rho_{AC}}{l_2} \qquad \text{(57)}$$

The change in the slope, ψ_{AC}, of any member, AC, is, therefore, equal to the length, ρ_{AC}, of the perpendicular, $C_a\ C'$, to AC, which locates C', divided by the length, l_2, of AC. ψ is positive when the member turns in a clockwise direction.

If, instead of a single triangle, there is a truss composed of a series of triangles, the process of determining ψ for each member for any given loading will be the same, that is, by first constructing a displacement diagram (which gives the ρ-values), and by dividing the values of ρ of each member by its corresponding length, l.

The graphical method is more expeditious than the analytical method; besides, the construction of a displacement diagram is a process well known to engineers. For this reason the graphical method of computing ψ, which has been commonly used by previous computers, will be used here. In constructing a displacement diagram of a truss for any given loading, assume an arbitrary point, a, and an arbitrary direction, $a\,b$, as fixed. This generally gives different vertical displacements, θ_A and θ_B, for A and B, the points of supports. For the calculations of the vertical displacements of the panel points, it is necessary to introduce a correction (as suggested by Mohr), in the Williot diagram. The supports, A and B, Fig. 19, do not move vertically. In order to satisfy this condition in the displacement diagram, the truss must be rotated as a whole about A' or B' by an amount,

$$\psi'' = \frac{(\theta_B - \theta_A)}{L} \qquad (58)$$

thereby bringing $A'B'$ parallel to AB.

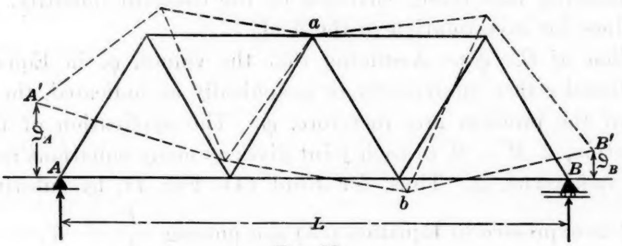


FIG. 19.

In the case of secondary stresses this is unnecessary. For the time being, however, suppose it is desirable to know the true angles, ψ and ϕ, that is, the angles which the chord and the tangent line to a member at its one end, after loading, make with the direction of the member before loading. It will then be necessary to have a displacement diagram in which no relative vertical displacement between A and B has occurred. Therefore, the diagram as a whole must be rotated about either A or B by an amount, ψ'', as given in Equation (58). All values, ψ', obtained by means of the displacement diagram must be corrected by this amount, constant for a given loading, giving $\psi = \psi' + \psi''$, the true changes in the slopes with reference to the original direction of the members. The next step depends on whether it is desired to obtain only the secondary stresses, or also the correct values of ψ and ϕ for purposes of experimental verification.

It is the writer's opinion that it is better to omit this correction, to solve for the bending moments, and, if these satisfy the fundamental equations—that the sum of the bending moments about each joint is zero—then to add

the constant correction to all the ψ's and ϕ's in order to obtain the true quantities, this procedure only in the cases where experimental verification is to follow.

The reason why this correction is unnecessary is simple. The bending moment at one end of a member is (see Equation (55)):

$$M_{nm} = \frac{2\,E\,I}{l} \cdot (2\,\phi_n + \phi_m - 3\,\psi_{nm})$$

Deducting the constant quantity, ψ'', from both the ϕ's and the ψ, does not affect M_{nm}, because the correction is $2\psi'' + \psi'' - 3\,\psi''$. In other words, it is also true that:

$$M_{nm} = \frac{2\,E\,I}{l} \cdot (2\,\phi_{n'} + \phi_{m'} - 3\,\psi'_{nm}).$$

Hence, the use of the uncorrected values of ψ, namely, ψ', would give a set of simultaneous linear equations for the ϕ's, and these would have the property that they, too, differ from the true ϕ's by the constant amount, ψ''. As the secondary stresses depend on the *M's*, it is, therefore, justifiable to use the ψ''s and ϕ''s.

In this investigation it is intended to omit the prime marks, and designate the uncorrected changes of slope as ψ, and the uncorrected angles of rotation as ϕ, remembering that these, corrected by the constant quantity, ψ'', would give the values for investigation in the field.

Calculation of the ϕ's.—Assuming that the values, ψ, in Equation (55) have been found either analytically or graphically as indicated, the only real unknowns of the problem are, therefore, ϕ. The application of the fundamental equation, $\Sigma\, M = 0$, to each joint gives as many equations as there are joints, and unknowns, ϕ. Thus, for Joint (4), Fig. 17, by substituting the values of M as expressed in Equation (55) and putting $\frac{I}{l} = K$,

$$\Sigma\, M_{4\xi} = 2\,E\,(2\,\phi_4\,\Sigma\,K_{4\xi} + \Sigma\,\phi_\xi\,K_{4\xi} - 3\,\Sigma\,\psi_{4\xi}\,K_{4\xi}) = 0$$

or,

$$2\,\phi_4\,\Sigma\,K_{4\xi} + \Sigma\,\phi_\xi\,K_{4\xi} - 3\,\Sigma\,\psi_{4\xi}\,.\,K_{4\xi} = 0 \ldots\ldots\ldots\ldots (59)$$

in which the numbers of the adjacent joints—in this case 2, 3, 5, 7, 6—are to be substituted for the subscripts, ξ.

The solution of the n simultaneous linear equations of the form of Equation (59) gives the values of the n unknowns, ϕ, where n denotes the number of joints. It is important to note that these are again of the type known as Gauss' normal equations and may be solved according to his method.

Solution of the Stresses.—The next step consists in substituting the values, ϕ and ψ, in Equation (59), thus giving the values of the moments, M, acting at each end of every member. Before finding the secondary stresses, verify the values of the moments by applying the check, $\Sigma\, M = 0$, at each joint.

The secondary stress is then readily obtained from,

$$f = \frac{M\,y}{I} = 2\,E\,I\,(2\,\phi_a + \phi_n - 3\,\psi_{an})\,\frac{y}{l\,I}$$

or by putting $E = 1$,

$$f = \frac{2\,y}{l} \cdot (2\,\phi_a + \phi_n - 3\,\psi_{an}) \ldots\ldots\ldots\ldots\ldots\ldots (60)$$

in which,

$\phi_a =$ the value of ϕ for the near end of the member;
$\phi_n =$ the value of ϕ for the far end of the member; and
$\psi_{an} =$ the change in the slope of the member.

(Note that the ψ's from the deflection diagram are regarded as E multiplied by the true ψ's. The resulting ϕ's are, therefore, also the true ϕ's multiplied by E. Hence, although E is always regarded as equal to unity it will still give the true secondary stresses.)

6.—Mohr's Elastic Weight Method

Small Movements of a Chain of Elastic Weights.—In Fig. 20, 1, 2, 3, 4 ...., represent the joints of a plane chain of members. These members are caused to move in the plane while Joint (1) remains fixed.

Let the members, 1–2, 2–3, 3–4, ..., have the lengths, l_1, l_2, l_3, l_4, ..., before any stresses are present, and let these lengths become $l_1 + d\,l_1$, $l_2 + d\,l_2$, $l_3 + d\,l_3$, $l_4 + d\,l_4$, ..., under loading. Then, $\frac{d\,l_1}{l_1}$, $\frac{d\,l_2}{l_2}$, $\frac{d\,l_3}{l_3}$, ..., are the strains, positive if the members are increased in length. Let these strains be denoted by λ_1, λ_2, λ_3, ..., that is,

$$\lambda_1 = \frac{d\,l_1}{l_1};\ \lambda_2 = \frac{d\,l_2}{l_2};\ \lambda_3 = \frac{d\,l_3}{l_3};\ \text{etc.,} \ldots\ldots\ldots\ldots (61)$$

The initial positions of the joints are determined by their distances, (u_1, v_1), (u_2, v_2), (u_3, v_3), ..., from two axes at right angles to each other. These axes may be chosen anywhere, but must fulfill one condition, namely, if the u-axis is once chosen in direction, the v-axis must be situated in such a way, that a clockwise rotation through 90° of the u-axis will make it coincide with the v-axis (Fig. 20). The co-ordinate of a point with regard to an axis is considered positive if, looking along that axis in its positive

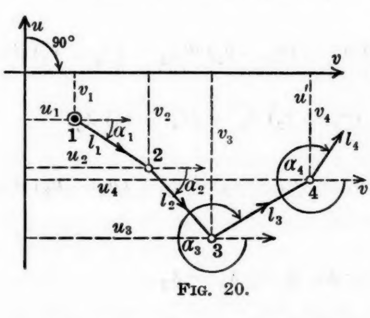


Fig. 20.

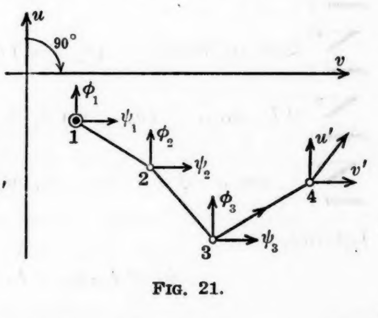


Fig. 21.

direction, the point lies on the right-hand side. Thus, all the co-ordinates in Fig. 20 are positive. The angles, α_1, α_2, α_3, ..., between the v-axis and the members, 1–2, 2–3, 3–4, ..., are the amounts of turning, in a clockwise direc-

tion, that would bring the v-axis into coincidence with the members, 1–2, 2–3, 3–4, ...

When $l_1, l_2, l_3, \ldots$, are changed by the small amounts, $d\,l_1, d\,l_2, d\,l_3, \ldots$, the angles, $\alpha_1, \alpha_2, \alpha_3, \ldots$, are changed by the corresponding small amounts, $d\,\alpha_1, d\,\alpha_2, d\,\alpha_3, \ldots$. Let,

$$\phi_1 = d\,\alpha_1,\ \phi_2 = d\,\alpha_2 - d\,\alpha_1,\ \phi_3 = d\,\alpha_3 - d\alpha_2 \ldots\ldots\ldots\ldots(62)$$

and, therefore,

$$d\,\alpha_1 = \phi,\ d\,\alpha_2 = \phi_1 + \phi_2,\ d\,\alpha_3 = \phi_1 + \phi_2 + \phi_3 \ldots\ldots\ldots\ldots(63)$$

The co-ordinates (u, v), of a joint, for example (u_4, v_4) of Joint (4), are expressed by,

$$\left.\begin{aligned} u_4 &= u_1 + l_1 \cos\alpha_1 + l_2 \cos\alpha_2 + l_3 \cos\alpha_3 = u_1 + \sum_1^3 l \cos\alpha \\ v_4 &= v_1 + l_1 \sin\alpha_1 + l_2 \sin\alpha_2 + l_3 \sin\alpha_3 = v_1 + \sum_1^3 l \sin\alpha \end{aligned}\right\} \ldots(64)$$

The small changes in the co-ordinates of Joint (4), due to the combined changes in the lengths of the members and the angles which the members make with the v-axis, are found by differentiating Equations (64). Keeping in mind that u_1 and v_1 are constant quantities,

$$\left.\begin{aligned} d\,u_4 &= \sum_1^3 d\,l \,.\, \cos\alpha - \sum_1^3 l \sin\alpha \,.\, d\,\alpha \\ d\,v_4 &= \sum_1^3 d\,l \,.\, \sin\alpha + \sum_1^3 l \cos\alpha \,.\, d\,\alpha \end{aligned}\right\} \ldots\ldots\ldots\ldots(65)$$

As $d\,l_1 = l_1 \lambda_1$, from Equations (61),

$$\cos\alpha_1 = (u_2 - u_1) \,.\, l_1,$$

from Fig. 20, and,

$$\sin\alpha_2 = (v_2 - v_1) \,.\, l_1$$

the summations in Equations (65) become,

$$\left.\begin{aligned} \sum_1^3 d\,l \,.\, \cos\alpha &= (u_2 - u_1)\,\lambda_1 + (u_3 - u_2)\,\lambda_2 + (u_4 - u_3)\,\lambda_3 \\ -\sum_1^3 l \sin\alpha \,.\, d\,\alpha &= -(v_2 - v_1)\,d\,\alpha_1 - (v_3 - v_2)\,d\,\alpha_2 - (v_4 - v_3)\,d\,\alpha_3 \\ \sum_1^3 d\,l \,.\, \sin\alpha &= (v_2 - v_1)\,\lambda_1 + (v_3 - v_2)\,\lambda_2 + (v_4 - v_3)\,\lambda_3 \\ \sum_1^3 l \,.\, \cos\alpha \,.\, d\,\alpha &= (u_2 - u_1)\,d\,\alpha_1 + (u_3 - u_2)\,d\,\alpha_2 + (u_4 - u_3)\,d\,\alpha_3 \end{aligned}\right\} (66)$$

Letting,

$$\psi_1 = \lambda_1,\ \psi_2 = \lambda_2 - \lambda_1,\ \psi_3 = \lambda_3 - \lambda_2$$

or,

$$\lambda_1 = \psi_1,\ \lambda_2 = \psi_1 + \psi_2,\ \lambda_3 = \psi_1 + \psi_2 + \psi_3 \ldots\ldots\ldots\ldots(67)$$

and using the relations expressed by Equations (62), and (63), Equations (66) take the following neat forms:

$$\left.\begin{aligned}
&\sum_1^3 d\,l \,.\, \cos\alpha = (u_4 - u_1)\,\psi_1 + (u_4 - u_2)\,\psi_2 + (u_4 - u_3)\,\psi_3 = +\,u_{4\psi}, \text{ say}\\
-&\sum_1^3 l \sin\alpha \,.\, d\,\alpha = -(v_4 - v_1)\,\phi_1 - (v_4 - v_2)\,\phi_2 - (v_4 - v_3)\,\phi_3 = -\,v_{4\phi}, \text{say}\\
&\sum_1^3 d\,l \,.\, \sin\alpha = (v_4 - v_1)\,\psi_1 + (v_4 - v_2)\,\psi_2 + (v_4 - v_3)\,\psi_3 = +\,v_{4\psi}, \text{ say}\\
&\sum_1^3 l\cos\alpha \,.\, d\,\alpha = (u_4 - u_1)\,\phi_1 + (u_4 - u_2)\,\phi_2 + (u_4 - u_3)\,\phi_3 = +\,u_{4\phi}, \text{ say}
\end{aligned}\right\} \quad (68)$$

Therefore, the changes in the co-ordinates of Joint (4), Equations (65), have the values:

$$\left.\begin{aligned}
d\,u_4 &= u_{4\psi} - v_{4\phi}\\
d\,v_4 &= v_{4\psi} + u_{4\phi}
\end{aligned}\right\} \quad \dots\dots\dots\dots (69)$$

Now, imagine Joint (1), Fig. (21), to be loaded with a positive weight, ϕ_1, parallel to the u-axis, and a positive weight, ψ_1, parallel to the v-axis; and Joint (2) to be loaded with a positive weight, ϕ_2, parallel to the u-axis, and a positive weight, ψ_2, parallel to the v-axis, etc. It will be seen that $u_{4\phi}$ of Equations (69) is the negative value of the sum of the moments of ϕ_1, ϕ_2, ϕ_3, about the axis, u', through Joint (4) parallel to the u-axis, while $v_{4\psi}$ is the negative of the sum of the moments of ψ_1, ψ_2, ψ_3, about the axis, v', through Joint (4) parallel to the v-axis. Further, if all the axes are turned in a clockwise direction through 90°, the u-axis and the u'-axis take the places and sense of the v-axis and the v'-axis, respectively, while the v-axis and the v'-axis take the places of the u-axis and the u'-axis, but are opposite to them in sense. Therefore, $u_{4\psi}$ of Equations (69) may be regarded as the negative of the sum of the moments of the ψ's about the new u'-axis, while $v_{4\phi}$ is the sum of the moments of the ϕ's, about the new v'-axis. The negative sign of $v_{4\phi}$ in the equation means that $-\,v_{4\phi}$ is the negative of the sum of the moments of the ϕ's, about the new v'-axis. Remembering that a positive weight on the right-hand side of an axis, and a negative weight on the left-hand side of an axis give positive moments, the argument becomes clear.

Therefore, considering two sets of positive weights, $(\phi_1\ \phi_2\ \phi_3)$ and $(\psi_1\ \psi_2\ \psi_3)$, perpendicular to each other and acting at the Joints (1), (2), and (3), as shown in Fig. 21, the negative of $d\,v_4$ in Equations (69) may be obtained by taking the sum of the moments of the ϕ's about an axis, u', parallel to the u-axis, and those of the ψ's about an axis, v', parallel to the v-axis. If both axes are turned 90° in a clockwise direction, the negative of $d\,u_4$ in Equations (69) is the sum of the moments of the ϕ's, about the new v'-axis, and those of the ψ's, about the new u'-axis.

It is important to note that the argument is unaltered if, instead of Joint (4), the origin is any other point rigidly connected with the member, 3-4.

Small Movements of a Closed Chain.—For a closed chain, 1, 2, 3..... n, 1 (Fig. 22), where Joint (1) is fixed in position and where the co-ordinates are as before, proceeding round the chain in the order 1, 2, 3, 4......n, 1, Equations (69) become:

$$\left.\begin{aligned}
d\,u_n &= u_{n\psi} - v_{n\phi}\\
d\,v_n &= v_{n\psi} + u_{n\phi}
\end{aligned}\right\} \quad \dots\dots\dots\dots (70)$$

for Joint (n), and the axes u', v', now pass through Joint (n). It is possible, however, to go further and reach Joint (1). Considering Joint (1), not as the start but as the end of the route, Equations (69) become:

$$\left.\begin{aligned} d\,u_1 &= u_{1\psi} - v_{1\phi} \\ d\,v_1 &= v_{1\psi} + u_{1\phi} \end{aligned}\right\} \quad \text{.........................}(71)$$

and the axes, u', v', now pass through Joint (1). As Joint (1), however, is fixed, $d\,u_1 = d\,v_1 = 0$ and Equations (71) are really.

$$\left.\begin{aligned} u_{1\psi} - v_{1\phi} &= 0 \\ v_{1\psi} + u_{1\phi} &= 0 \end{aligned}\right\} \quad \text{.........................}(72)$$

If Joint (n) had been the terminus, it would have been necessary to consider all the ψ's, from ψ_1 to ψ_{n-1}, and all the ϕ's, from ϕ_1 to ϕ_{n-1}, in Equations (70). Proceeding to Joint (1) and obtaining Equations (72) adds ψ_n to the list of ψ's, and ϕ_n to the list of ϕ's. In other words, Equations (72) are true

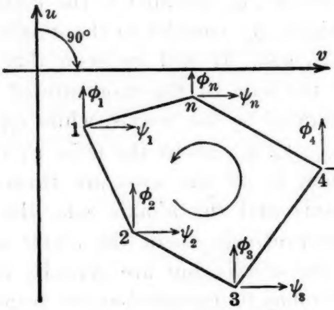


FIG. 22.

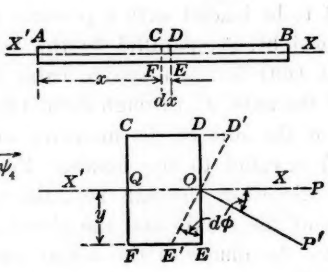


FIG. 23.

for all the weights, ψ and ϕ, of all the joints in the chain. Turning to Equations (62) and (67), it is evident that until now the ϕ's and ψ's have been taken as follows:

$$\left.\begin{array}{llll} \phi_1 = d\,\alpha_1 & \text{and} & \psi_1 = \lambda_1 & \text{for Joint (1)} \\ \phi_2 = d\,\alpha_2 - d\,\alpha_1 & \text{and} & \psi_2 = \lambda_2 - \lambda_1 & \text{for Joint (2)} \\ \phi_3 = d\,\alpha_3 - d\,\alpha_2 & \text{and} & \psi_3 = \lambda_3 - \lambda_2 & \text{for Joint (3)} \\ \dots\dots & & \dots\dots & \dots\dots \\ \phi_n = d\,\alpha_n - d\,\alpha_{n-1} & \text{and} & \psi_n = \lambda_n - \lambda_{n-1} & \text{for Joint } (n) \end{array}\right\} \text{....}(73)$$

The moments of all the weights about the axes, u', v', through Joint (1) are under consideration; the moments of ϕ_1 and ψ_1, which act at Joint (1), are, therefore, zero. In other words, Equations (72) are not affected if either the axes, ϕ_1, ψ_1, are neglected or some other values substituted for ϕ_1 and ψ_1. A study of Equations (73) would suggest a completion of the cyclic order, by making,

$$\phi_1 = d\,\alpha_1 - d\,\alpha_n \text{ and } \psi_1 = \lambda_1 - \lambda_n$$

This gives two new relations, for on adding all the ϕ's,

$$\Sigma\,\phi = 0 \text{.................................}(74)$$

and adding all the ψ's,

$$\Sigma\,\psi = 0 \text{.................................}(75)$$

Now, the set of ψ's is a horizontal one, while the set of ϕ's is a vertical one, therefore, each set resolves itself into a couple and as the moment of a couple is constant, no matter where the axis is taken, the origin may be chosen anywhere. The direction of the u-axis is arbitrary, but, once chosen, it controls the position of the v-axis as that into which the u-axis falls when it is turned 90° clockwise. In that case, Equations (72) hold, as well as Equations (74) and (75). These are four linear relations that exist between the $2n$-quantities, ϕ and ψ. Hence, if $(2n\text{-}4)$ of the $2n$-quantities, ϕ and ψ, of a closed chain are known, the remaining four may be found.

A Beam Subjected to Bending.—Consider a beam, AB, Fig. 23, and let it be subjected to bending. Consider a small length, dx, of the beam, distant x from A, magnified as shown. Before bending, its shape is $C\,D\,E\,F$; after bending, it becomes $C\,D'\,E'\,F$, $C\,D$ suffering an elongation, $D\,D'$, and $E\,F$ a contraction, $E\,E'$, while the neutral axis remains unchanged. Imagine a perpendicular, $O\,P$, rigidly attached to the face, $D\,E$; that perpendicular will be carried to the position, $O\,P'$, when the beam is bent. The change in angle between $O\,P$ and $O\,P'$, $d\,\phi$, is what has been regarded as a weight in the foregoing discussion. If $d\,x$ becomes infinitesimally small, $Q\,O$ may be regarded as one of an infinite number of links, while the $d\,\phi$'s at their right-hand ends, O, become what are known as "elastic weights", and, as is well known,

$$d\,\phi = \frac{M_x\,.\,d\,x}{E\,I} \quad \text{.......................(76)}$$

in which M is the bending moment at the section, x.

A Beam Subjected to a Bending Moment at Each End.—Consider a beam, 1-2, Fig. 24. Let a counterclockwise (positive) moment, M_{12}, act at End (1); similarly, let a positive moment, M_{21}, act at End (2). The shape into which

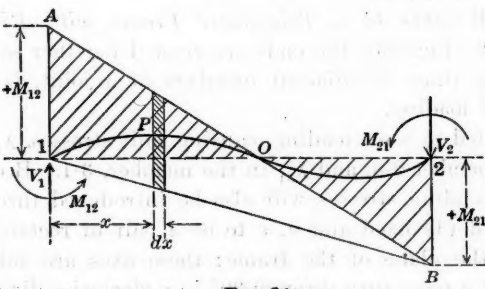


FIG. 24.

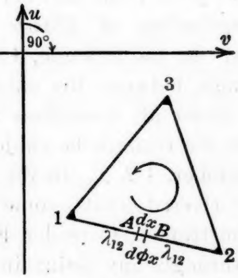


FIG. 25.

the beam is bent is somewhat as shown in the diagram. These moments induce a shear, V_1, upward at End (1), and a shear, V_2, downward at End (2). Taking moments about any point, P, distant x from End (1),

$$M_x = M_{12} - V_1 x$$

This proves that the variation of the moment is a linear one. Somewhere, however, between End (1) and End (2), there is a point of inflection, marked O, where the bending moment is zero. Hence, the bending moment diagram is as shown (shaded). Clearly, the bending moment area, $A\,O\,1$ is positive,

while $B\,O\,2$ is negative. If the portion, $1\,O\,B$, were added to $A\,O\,1$, to make a positive area, $A\,B\,1$, and also the same portion, $1\,O\,B$, to $B\,O\,2$, to make a negative area, $1\,2\,B$, these two newly found areas would have the same effect as the two original areas.

From Equation (76), however,

$$d\,\phi = \frac{M_x\,dx}{E\,I}$$

which represents the small bending moment area at x of width, $d\,x$. E is constant and the member is assumed to have a constant moment of inertia, I. Now, regard the positive bending moment area, $AB1$, as being replaced by all its small elastic weights, $d\,\phi$; these weights can themselves be replaced by their sum, acting at their centers of gravity, before treating them in any moment problem; but,

$$\phi_{12} = \int d\,\phi \text{ from End (1) to End (2),}$$

$$= \int_0^{l_{12}} \frac{M_x\,dx}{E\,I}$$

$$= + \frac{M_{12} \cdot l_{12}}{2\,E\,I}.$$

That is, it is permissible to replace all the elastic weights, $d\,\phi$, belonging to the positive area, $AB1$, by one elastic weight the value of which is ϕ_{12}, acting at the first "third" point from End (1) to End (2).

Similarly, all the elastic weights belonging to the negative area, $12B$, may be replaced by one elastic weight, $\phi_{21} = -\dfrac{M_{21} \cdot l_{12}}{2\,E\,I}$, acting at the second "third" point from End (1) to End (2).

Application of Elastic Weights to a Triangular Frame with Riveted Joints.—In the triangle, 1 2 3 (Fig. 25), the ends are riveted together so that the angle between the elastic lines of adjacent members at a joint, is constant under all conditions of loading.

Let the triangle be subjected to some loading, causing unit stresses, λ_{12}, in the member, 1-2, λ_{23} in the member 2-3, and λ_{31} in the member, 3-1. Because of the riveted joints, some bending stresses will also be introduced throughout the frame. Consider End (1) fixed and u, v to be a pair of rectangular axes through any point in the plane of the frame; these axes are subject, however, to the condition that u must turn through 90° in a clockwise direction to coincide with v. It is assumed that anywhere in a member the direction of the displaced direct stress, due to the bending effect, deviates so little from that of the line joining the ends of the member, that the direction of the direct stress coincides with that line.

The triangle now becomes a closed chain, composed of an infinite number of links, dx, the joints, (1), (2), and (3), of the triangle serving as joints between adjacent links of two adjacent members. The riveted connections make this continuity possible. Following round the chain in a counterclockwise direction, take any one of the links of the member, 1-2, except the first,

such as AB, in Fig. 25. Joint (A) has a unit stress, λ_{12}, likewise Joint (B). Thus, in Equations (73), as applied to a closed chain, $\psi_B = 0$, while ϕ_B is the small elastic weight, $\frac{M_B\,dx}{E\,I}$, where M_B is the bending moment in the member at B, etc. If all the links but the first are taken, all the ψ's are zero while all the ϕ's are present.

In connection with the first link of the member, 2-3, the λ first met is the λ_{12}, while the second λ is λ_{23}. Therefore, at a point on the member, 2-3, adjacent to Joint (2), the ψ is $\lambda_{23} - \lambda_{12}$, and as the link is infinitesimally small, $\lambda_{23} - \lambda_{12}$ may be regarded as acting at Joint (2). Here, however, the ϕ is zero, as the end tangents to the elastic lines at Joint (2) are assumed to retain their relative positions. In considering the remaining links of the member, 2-3, all the ψ's are again zero, while the ϕ's are the elastic weights due to the bending of the member, 2-3.

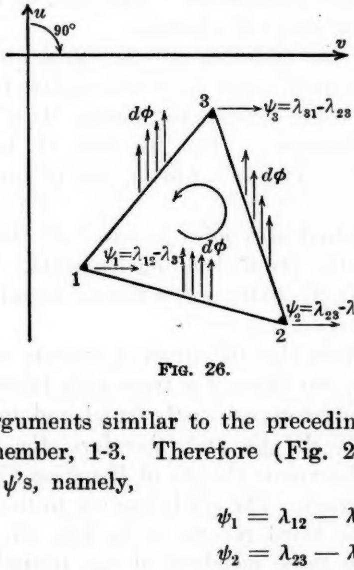


FIG. 26.

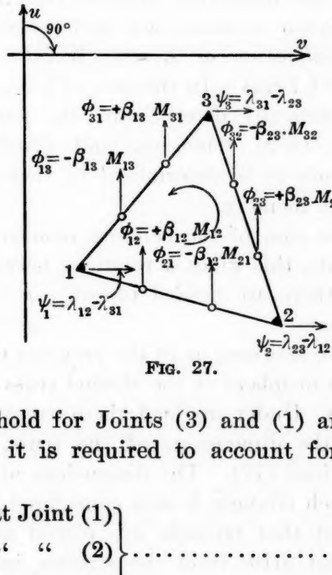


FIG. 27.

Arguments similar to the preceding, hold for Joints (3) and (1) and for the member, 1-3. Therefore (Fig. 26), it is required to account for only three ψ's, namely,

$$\left.\begin{aligned} \psi_1 &= \lambda_{12} - \lambda_{31} \text{ at Joint (1)} \\ \psi_2 &= \lambda_{23} - \lambda_{12} \text{ `` `` (2)} \\ \psi_3 &= \lambda_{31} - \lambda_{23} \text{ `` `` (3)} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (77)$$

while there is an infinite number of $d\,\phi$'s along the members, 1-2, 2-3, and 3-1. It was proved in the preceding discussion that the infinite number of elastic weights, $d\,\phi$, of the member, 1-2, can be replaced by one elastic weight $= + \frac{M_{12}\,l_{12}}{2\,E\,I}$, acting at the first "third" point from the member, 1-2, and one elastic weight $= - \frac{M_{21}\,l_{12}}{2\,E\,I_{12}}$, acting at the second "third" point from the member, 1-2, in which $+ M_{12}$ is the bending moment at End (1), and $+ M_{21}$ is the bending moment at End (2), of the member, 1-2.

A similar argument holds for the members, 2-3 and 3-1. It is important to remember, that, as in the analysis of a closed chain, it is necessary to go

round the circuit in one direction. In Fig. 26, this direction is counterclockwise, hence, the first elastic weight met along the member, 1-2, is $+\frac{M_{12}\, l_{12}}{2\, E\, I_{12}}$, while the second is $-\frac{M_{21}\, l_{12}}{2\, E\, I_{12}}$, etc.

For convenience, let,

$$\beta = \frac{l}{2\, E\, I} \qquad (78)$$

The problem finally becomes a question of establishing enough relations between the ψ's and ϕ's, shown in Fig. 27. It is assumed that the direct stresses in the riveted frame are the same as if it were pin-connected; the λ's, and, therefore, the ψ's, are assumed to be known. There are six unknown bending moments, M, contained in the ϕ's. Assuming three of these bending moments to be known, moments of all the weights about three sets of axes, their origins not being collinear, will give three equations which will determine the remaining three unknown bending moments. The axes, of course, are chosen in such a way as to obtain the simplest relations.

Application of Elastic Weights to the Solution of the Moments in a Riveted Truss.—In the case of a riveted truss, made up of triangular elements and statically determinate, the joints being frictionless hinges there are n joints, (n-2) triangles, and (2n-3) members. The number of bending moments to be determined is, therefore, $2\,(2n\text{-}3) = (4n\text{-}6)$, one of each end of each member.

The sum of the bending moments found at a joint is zero. As there are n joints, this gives n relations between the (4n-6) bending moments. Therefore, there are needed $(4n\text{-}6) - n = 3\,(n\text{-}2)$ additional relations to solve the problem.

One assumes, as in the previous methods, that the direct or primary stresses in the members of the riveted truss are also those of a truss with frictionless hinges. Under any load, these stresses, therefore, are easily found, and, together with the dimensions of the truss, give the λ's and, therefore, the ψ's of Equations (77). The dimensions also determine the β's of Equation (78).

Each triangle is now considered by itself. The ϕ's belonging to the members of that triangle are placed at the third points, as in Fig. 27. The ψ's that arise from the stresses in the three members of the triangle, are placed at the vertices. It should be noted that the ϕ's of a side remain the same, in magnitude and sign, for the two triangles to which that side is common. Hence, after following round the first triangle in a clockwise direction, it is necessary, in order to preserve the signs of the ϕ's, to go round the next triangle in a counterclockwise direction, etc. The ψ's and ϕ's are regarded as weights. As each triangle is a closed chain, the sum of the bending moments of all the ϕ's about one axis, and the bending moments of all the ψ's about a perpendicular axis, is zero, the condition for the two axes being that the ϕ-axis must turn through 90° in a clockwise direction in order to coincide with the ψ-axis. Three such sets of axes are chosen for each triangle, to give the three relations required between the ϕ's and ψ's of each triangle. The three origins must not be collinear.

The method of procedure is first to assume the end moments of one of the two members making the first joint to be known. However, the sum of the bending moments at a joint is zero, therefore, that bending moment at the joint of the remaining member is also known. The remaining three bending moments of the first triangle are then solved by choosing three sets of axes. This being done in a definite way, as will be shown in the example, each additional equation will contain only one new unknown bending moment, all the others having already been expressed in the earlier equations. It must be borne in mind that as soon as the stage is reached where all but one of the bending moments around a joint are expressed in the equations already found, the relation, $\Sigma M = 0$, for the joint is introduced. It will be found that the $\Sigma M = 0$'s for the last two joints introduce no fresh bending moments; their use is found later. Call this whole set of equations, ($2n$-6) in number, Set No. 1.

The next step is to express every M of Set No. 1 in terms of the M's found previously. Call this new set, Set No. 2.

It is then a matter of substitution to change Set No. 2 into Set No. 3, where every M is expressed in terms of the two M's assumed to be known at the start. Here, the last two equations of Set No. 1 are of value, for they are found to be two simultaneous linear equations with the two assumed M's which, therefore, are found. Substituting in Set No. 3 gives the values of all the M's from which to determine the secondary stresses. The method is easy to follow, as will be shown in an example.

7.—Mao's Methods

Fundamental Principles

The Fundamental Conception.—Quoting from Dr. Mao's thesis (28): Secondary stresses are fiber stresses, produced in the members, due to bending moments developed around the joints. There are many sources from which the moments are derived, but the principal one is the rigidity of connections. If $M =$ the bending moment, $I =$ the moment of inertia of the member, $y =$ the distance from neutral axis to the fiber, the stress of which is required, and $f =$ the secondary stress, then,

$$f = \frac{M\,y}{I} = \frac{M}{S},$$

in which S is the section modulus. Therefore,

$$M = S\,.\,f \qquad (79)$$

In other words, if S is assumed as a force, and M is the moment it produces about a point, then the offset of the force from the point gives the value of the secondary stress.

Let AB be a member, connecting the Joints (A) and (B), and S its section modulus (Fig. 28). If S is assumed to be a force acting in this member, like the primary stress, it will produce moments at Joints (A) and (B), if its line of action is displaced from the axis of AB, and the offsets, f_a and f_b, will then be the stresses produced at Joints (A) and (B) (Equation (79)).

Particularly, if the moments about Joints (A) and (B) are those due to rigidity of joints, the offsets, f_a and f_b, will be the secondary stresses. The line of action of S will be called hereafter the "Secondary Stress Line" or briefly "Stress Line", no confusion being caused thereby, as the primary stress line is not shown or is understood to be the axis of the member.

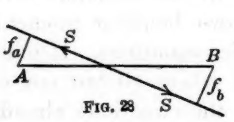


FIG. 28

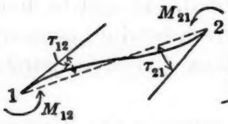


FIG. 30.

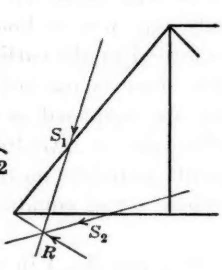


FIG. 29.

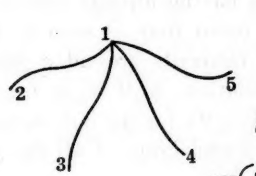


FIG. 31.

FIG. 32

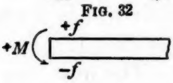


FIG. 33.

It is now evident that the values of f depend only on the location of the stress line, S, while the moment, M, will also depend on its magnitude. As the magnitude of S is constant, being the section modulus, the position and direction of S, and, consequently, f, will depend solely on M.

Let this conception be extended to every member of the truss. There will then be as many stress lines as there are members. Each stress line will be displaced to an extent proportionate to the bending moments produced. To utilize this fact, by reversing the process, it is at once evident that, if the stress lines are located so as to satisfy various imposed conditions, the offsets of the stress lines will give directly the secondary stresses. This conception is fundamental.

The first condition, to which the various stress lines are subjected, is that the total resulting moments around every joint must be zero. (Eccentric moments are excluded here, but could be taken care of very readily.) Graphically, this means that, if a force diagram is drawn of all the stress lines in the truss, and an equilibrium polygon is likewise constructed on the truss diagram, these two polygons must, respectively, close. This, however, is impossible for the stress lines adopted previously, because, the values of S being constant, they could not be made to balance each other. Thus, in Fig. 29, the stress lines, S_1 and S_2, could not balance each other, unless they are on the same straight line, which is impossible. To overcome this difficulty, an external ideal force will be applied to the joint, with such magnitude and direction that the equilibrium is maintained around the joint. Thus, in Fig. 29, a force, R, may be introduced which will balance the forces, S_1 and S_2. In general, the magnitude and direction of R is equal to the resultant of the stress lines acting on the joint, but with opposite sense. It is important to notice that the position of R must be such that its line of action passes through the joint under consideration, so that there will be no external moment. In case of eccentric connections, the position of R may be adjusted so that its value and the offset from the joint will give the eccentric moment.

As a convenience in terminology, the term, "stress line", will be understood to be the internal secondary stress line, S, while the external force will be understood as the balancing force, R.

With the conceptions of S and R thus established, it is now possible to draw a force diagram around every joint from which to construct an equilibrium polygon. If these two polygons are made to close in any given case, the first condition, that the sum of internal moments must be zero around any joint, is everywhere satisfied. Further, from the condition that the total external moment is zero for all the joints of the truss, it may be inferred that both the force and equilibrium polygons of the external ideal forces, R, must, respectively, close.

The Fundamental Equation.—Dr. Mao's method for establishing this relation is unnecessarily complicated; an easier way, therefore, will be adopted as follows:

It has been shown (Fig. 5 and Equations (37) and (38)) that, if a member, 1-2, Fig. 30, is subjected to two positive bending moments (counterclockwise being the positive sense), M_{12} and M_{21}, at its ends, and if τ_{12} and τ_{21} are the angles between the end tangents and the chord at Member 1 and Member 2, respectively, then,

$$\tau_{12} = \frac{l_{12}}{6\,E\,I_{12}}\,(2\,M_{12} - M_{21})$$

and,

$$\tau_{21} = \frac{l_{12}}{2\,E\,I_{12}}\,(2\,M_{21} - M_{12})$$

Now, imagine a number of members, 1-2, 1-3, 1-4, 1-5, entering Joint (1), (Fig. 31),

$$\tau_{12} = \frac{L_{12}}{6\,E\,I_{12}}\,(2\,M_{12} - M_{21})$$

$$\tau_{13} = \frac{L_{13}}{6\,E\,I_{13}}\,(2\,M_{13} - M_{31})$$

Therefore,

$$\tau_{12} - \tau_{13} = \frac{L_{12}}{6\,E\,I_{12}}\,(2\,M_{12} - M_{21}) - \frac{L_{13}}{6\,E\,I_{13}}\,(2\,M_{13} - M_{31})$$

However, $\tau_{12} - \tau_{13} = \Delta\,\alpha_{213}$, this being the change in the angle, 213, under the loading, considering the joints as frictionless hinges. (See Fig. (6) and Equation (27)). Therefore,

$$L_{12}\left(\frac{2\,M_{12}}{I_{12}} - \frac{M_{21}}{I_{21}}\right) - L_{13}\left(\frac{2\,M_{13}}{I_{13}} - \frac{M_{31}}{I_{31}}\right) = 6\,E\,\Delta\,\alpha_{213}$$

or, by using Equation (79),

$$\frac{L_{12}}{y_{12}}\,(2\,f_{12} - f_{21}) - \frac{L_{13}}{y_{13}}\,(2\,f_{13} - f_{31}) = 6\,E\,.\,\Delta\,\alpha_{213}$$

In general, if two members, l-m and l-n, meet in a joint, Fig. 32, and if the order of consideration is in a counterclockwise direction,

$$\frac{L_{lm}}{y_{lm}}\,(2\,f_{lm} - f_{ml}) - \frac{L_{ln}}{y_{ln}}\,(2\,f_{ln} - f_{n}) = 6\,E\,.\,\Delta\,\alpha_{mln} \quad \ldots\ldots\ldots (80)$$

in which,

$L =$ length of the member;

$y =$ extreme fiber distance;

$f_{lm} =$ secondary stress of the member, l-m, at the end, l; and

$\Delta\, \alpha_{nlm} =$ change of angle between the members, l-n and l-m.

This is Dr. Mao's "Fundamental Equation" which, he says, forms the basis for his "Theory of Deformation Contour" (see page 1016).

Analytical Equations

In taking the bending moment as positive when it acts in a counterclockwise direction, it is seen that f, the stress of the fiber first met, is negative while that of the one passed over last in this counterclockwise route, is positive (Fig. 33).

On substituting U for $\frac{L}{y}$ and K for $6\, E\, .\, \Delta\, \alpha$, Equation (80) is changed to,

$$U_{lm}\,(2\, f_{lm} - f_{ml}) - U_{ln}\,(2\, f_{ln} - f_{nl}) = K_{mln}$$

or,

$$2\, f_{lm} - f_{ml} = \frac{U_{ln}}{U_{lm}}\, 2\, f_{ln} - f_{nl}) + \frac{K_{mln}}{U_{lm}} \ldots\ldots\ldots\ldots (81)$$

Now, let,

$$r_{lm} = 2\, f_{lm} - f_{ml}$$

and,

$$r_{ml} = 2\, f_{ml} - f_{lm} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (82)$$

Equation (81) can then be written:

$$r_{lm} = \frac{U_{ln}}{U_{lm}}\, .\, r_{ln} + \frac{K_{mln}}{U_{lm}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (83)$$

From Equation (82),

$$r_{lm} = 3\, f_{ml} - 2\, r_{ml} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (84)$$

and also,

$$f_{ml} = \frac{1}{3}\,(2\, r_{ml} + r_{lm}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (85)$$

This shows that the value of r_{lm} may be obtained from the values of f and r at the other end of the member, without knowing the value of f_{lm}.

Graphical Constructions

Preliminary Considerations.—Quoting again from Dr. Mao's thesis:

Scales.—For graphical constructions, the scale of f is evidently dependent on that of $E\, \Delta\, \alpha$. From the fundamental Equation (81), it is seen that the scale of f is of the same dimensional degree as that of $\frac{K_{mln}}{U_{lm}}$, or of $\left(\frac{6\, E\, \Delta\, \alpha}{\frac{L}{y}}\right)_{lm}$.

As L is always expressed in feet and y in inches, L and y may be represented by the same unit, if $6\, E\, \Delta\, \alpha$ is changed into $\frac{E\, \Delta\, \alpha}{2}$. Hence, hereafter, K is to

be understood as $\frac{E \Delta \alpha}{2}$, with the understanding that y is expressed in inches, but represented as if it were in feet. As the values of $E \Delta \alpha$ are found from the primary stresses, by graphic or analytic methods, it is convenient to consider only one-half the primary stresses. In graphic methods, the scale of primary stresses may be made only one-half as large as that used for measuring the values of $E \Delta \alpha$. This will give the value of K directly.

The scale of S, the section modulus, used in constructing the force diagrams, may be any value independent of f, as it is only the direction that is required. The scale used for r must be the same as that for K or f.

Signs.—In the present work there are two instances where a conception of signs is necessary. One is rotational and the other is linear. The first applies to moments, angles, circular sections, and directions of contours, while the second applies to the stress lines, their offsets, representations of f and r, and the co-ordinates of variables. The rotational direction will always be considered positive if it is counterclockwise. For linear directions there are many considerations, but the principle is, with the exception of the stress lines, that if ab is the base of a straight line through a and b, with a as origin, the distance from a to any point, x, on ab, is positive, if x is laid off from a toward b, and negative, if away from b. That is, it is positive if it is between a and b, and negative if it is on the side of a away from b.

For stress lines the following convention is necessary: It has been shown that a positive moment is that, which will rotate a member in a counterclockwise direction; as the moment is represented by the product of f and S, the direction of S should be such that a counterclockwise moment would give a positive stress, f. For the sake of uniformity, all the stress lines, S, will be considered as compressive stresses in the members. Under this convention, if the stress lines in the members, 1-2 and 1-3, are as shown, Fig. 34, f_{12} and f_{21} will be negative (moment is clockwise about Joint (1) and also about Joint (2)), and f_{13} and f_{31} will be positive (moments about Joints (1) and (3) are counterclockwise). Hence, when the position of the stress line

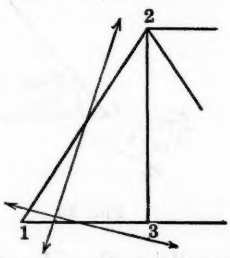


FIG. 34.

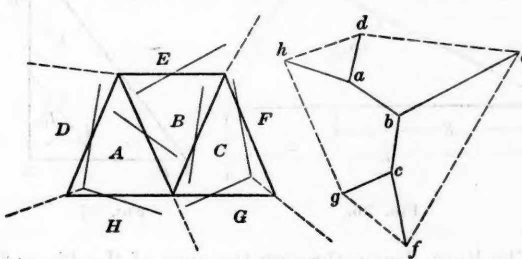


FIG. 35.

is known, both the sign and the magnitude of f are at once obtained. Further, by this convention, the force diagram of stress lines, S, assumes a definite form, which tends to facilitate its construction. Consider the simple truss, as shown in Fig. 35. The letters, a, b, and c, in the force diagram correspond to the spaces, A, B, and C, of the truss diagram. It will be seen that the forces da, ab, bc, and cf, may be considered, for illustration, as the trunk of

a tree, while the forces, *ah, bc,* and *cg,* are its branches. This makes it easy to remember that all the stress lines in web members and end posts form the trunk of a tree, while those of the chord members form the spreading branches.

Truss Diagrams.—Two or three truss diagrams are necessary in the graphic methods presented in this paper. One of the diagrams will be used to record given information and to construct the values of K. The use of this diagram will be described, while the construction on the remaining diagrams will be considered subsequently.

The quantities, S, and primary stresses, in pounds per square inch, are first marked on all the members of the truss. The "y-circles" are then constructed. These are the circles, with radii equal to the y's of the members, drawn at one end of each member. Let L be the length of the member, in feet, and y, in inches (Fig. 36). At one end of the member draw a circle with radius equal to y, expressed in inches. The scale used is the same as that for L, which is in feet, the value of y being thus magnified twelve times. From the other end of the member draw a line tangent to this circle. This line will be known as the "U-line." To find the value of $\frac{K}{U} = \frac{\frac{K}{L}}{y}$, lay off a segment equal to K from the end of the member, from which the U-line is drawn. At the end of the segment, K, draw a circle, tangent to the U-line. The radius of this circle then gives the value required.

Changes of Angle.—The method of Ritter (page 977) is followed. The purely graphical construction of Mao is dangerous, because of possible mistakes in signs. All the triangles of the truss to be considered have a right angle; let P_{13}, P_{23}, and P_{12} (Fig. 37) be the stresses in the sides, 1-3, 2-3, and 1-2, respectively. These are drawn, to a certain scale, from the opposite vertices perpendicularly toward the sides, 1-3, 2-3, and 1-2, respectively.

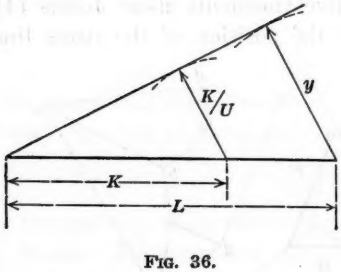


FIG. 36.

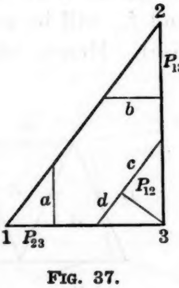


FIG. 37.

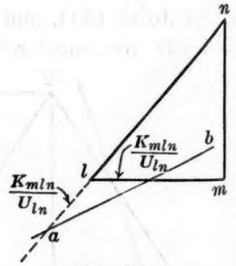


FIG. 38.

The lines, drawn through the ends of the P's and parallel to the sides opposite, give the quantities, a, b, c, and d. Now, the change of angle at Joint (2) is given by $(b\text{-}d)$. If P_{13} and P_{12} are both positive, it would be possible to superimpose d on b graphically, and thus mark off a length of line (positive if b is greater than d, and *vice versa*) that represents the change of angle, $\Delta\,\alpha_{123}$. It is readily seen that, with the P's differing in sign, such a process is extremely troublesome. It is wiser to introduce Mao's analytical method of

obtaining the changes of angle into his graphical solutions. This tabulation will be shown later.

Deformation and Property Lines.—Consider Equation (83):

$$r_{lm} = \frac{U_{ln}}{U_{lm}} \cdot r_{ln} + \frac{K_{mln}}{U_{lm}}$$

the quantities, $\left(\frac{U_{ln}}{U_{lm}}\right)$ and $\left(\frac{K_{mln}}{U_{lm}}\right)$, are constants under the particular loading of the truss, whereas r_{lm} and r_{ln} may be regarded as two unknowns. There is, therefore, a linear relation between them.

If lm and ln (Fig. 38) be the two axes (oblique in general), mark off $+\frac{K_{mln}}{U_{lm}}$ along $l\,m$ and $-\frac{K_{mln}}{U_{lm}}$ along $l\,n$; this gives the straight line, a-b, which is called the "property line" of the triangle for the angle, mln. It is, in fact, the graph of Equation (83) on the two members considered as axes. There are three such deformation lines for every triangle. With regard to the convention of drawing them, it must be remembered that the U's are positive quantities, and the K's can be either positive or negative. If the K of an angle is positive, lay off a positive quantity along the member first met in a counterclockwise direction, and a negative quantity along the second member; if the K is negative, lay off a negative quantity along the first member and a positive quantity along the second.

Dr. Mao introduces what he terms the "property lines". From the fact that the sum of the three angles of a triangle remains constant, $\Sigma K = 0$ for every triangle. Using this, it can be shown analytically, that the three straight lines, through the vertices of a triangle and parallel to the corresponding deformation lines, are concurrent. Such a line would be:

$$r_{lm} = \frac{U_{ln}}{U_{lm}} \cdot r_{ln} \quad \text{.......................(86)}$$

the property line for the vertex, l, of the triangle, mln (Fig. 38). These lines may be regarded as the deformation lines for zero loading, when all the K's are zero. Use is made of this property to check the drawing of the deformation lines.

For the construction of the property and deformation lines, proceed as follows: Consider the truss diagram, Fig. 39, with all the U-lines drawn. Describe an arc of a circle with as large a radius as it is convenient to take and with Joint (1) as the center. Let this arc cut the member, 1-2, in a and the member, 1-3, in b. Draw perpendiculars to U_{12} and U_{13} from a and b, respectively. Let these be u_{12} and u_{13}.

In the second diagram, Fig. 40, lay off u_{13} from Joint (1) along the member, 1-3. Call the point, c. From c, lay off the length, $cd = u_{12}$, along the line through c, parallel to the member, 1-2. Join 1 to d. This is the property line for the angle $A1$, as $\frac{u_{12}}{u_{13}} = \frac{U_{12}}{U_{13}}$, fulfilling the requirements of Equation (86). This is repeated for every angle in the truss. In order to obtain the best results

the quantities, u_{12}, u_{13}, and, therefore, the radius of the circle, $a\,b$, must be as large as possible.

For the deformation lines, instead of any radius, select K_{312} to the scale of the f's, and describe an arc of a circle about 1 as the center. As the whole solution depends on the accuracy of locating these deformation lines, special care must be taken in this part of the work. Let the new points of intersection in Fig. 39 be a' and b', and the corresponding perpendiculars, u_{12}' and u_{13}'. Now, lay off u_{13}' along the member, 1-3, and u_{12}' along the member, 2-1, produced, if K_{312} is positive, whereas, lay off u_{13}' along the member, 3-1, produced, and u_{12}' along the member, 1-2, if K_{312} is negative. This is shown in Fig. 41 for convenience; in practice, the construction of the two sets of lines is on one diagram. A study of Equation (83) will prove that the line, ef, is the deformation line of the angle, $A1$. These deformation lines. now, must be parallel to the corresponding property lines, of which those belonging to a triangle must be concurrent.

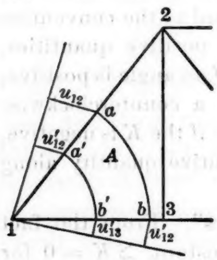


FIG. 39.

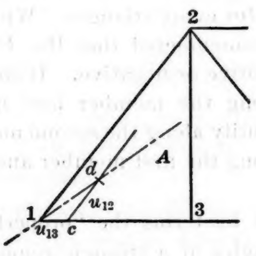


FIG. 40.

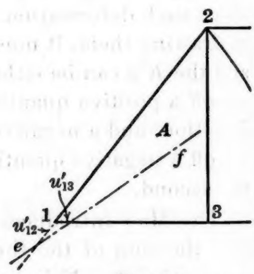


FIG. 41.

The Equilibrium Contour.—Quoting again from Dr. Mao's thesis: Under "Fundamental Conceptions" (page 1008) (Fig. 29), it has been shown, that the stress lines, S, could be made to balance each other around any joint by the introduction of an ideal external force, R. In the process of construction, however, the conception of stress lines as there adopted is not convenient, as it would involve a different force diagram for each change of the stress lines, which is unavoidable in the graphical construction. A speedy method demands that one force diagram shall serve the whole process, that is, the direction of the stress lines is to be kept constant. This fixed direction of the stress lines is best chosen perpendicular to the axis of the member, for then the offset, $(=f)$, would be found along the axis of the member. For example, let S_{12}, Fig. 42, cut the member, 1-2, in c. The stress, f_{12}, is then equal to 1-c, positive if c is between 1 and 2, and negative if on the line, 2-1, produced. For the end, 2, the same force, S_{12}, should be shifted to a point the distance of which from the end, 2, gives the stress, f_{21}. By this modification, the direction of stress lines, S, is made constant for the members, and one force diagram is sufficient for different values of f at the same end of the same member. Here, these S lines are no longer the "Stress Lines", in the correct use of the term, but they will be known as the "Equilibrium Lines".

The direction of the equilibrium lines should be such, that they give a counterclockwise moment around the joint when the values of f are positive,

that is, when the lines cut the members somewhere between the joints. Thus, the direction of S_{12} should be upward, whether it is on the right-hand or the left-hand side of Joint (1). If on the left side of Joint (1), the moment is clockwise, and f is then negative.

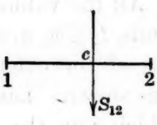


FIG. 42

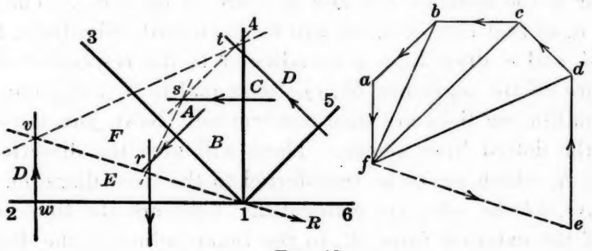


FIG. 43.

Consider the joint of a truss, as shown in Fig. 43. The force diagram is assumed to have been constructed. If the stresses in the members, 1-2, 1-3, 1-4, and 1-5, are known, the positions of the equilibrium lines are known. Then the moment of the member, 1-2, equals the force, *af*, multiplied by the perpendicular distance, f_{12}. If f_{12} is negative, it should be laid off on the side of 1 remote from 2, that is, along the member, 1-6. The closing line, *fe*, furnishes the external force, *R*, at the joint. The direction of the force is constant, and *R* may be drawn on the diagram as a part of the truss.

Suppose that the stress in the member, 1-6 ($= f_{16}$) is required, that is, the location of the force, *de*, the direction of which is upward, found by passing around the joint in a counterclockwise direction. In the force diagram, take f as the pole, and draw the rays, *fb, fc,* and *fd.* At the intersection, *r*, of the forces, *FA* and *AB*, draw a line parallel to *fb* meeting the force, *BC*, at *s*. From *s* draw a line parallel to the ray, *fc*, meeting the force, *cd*, in *t*. From *t*, draw a line parallel to the ray, *fd*, meeting the force, *FE*, that is, *R*, at *v*. Then, the force, *DE*, must pass through *v*. Therefore, draw a perpendicular from *v* on the side, 1-6, and the distance 1*w* will give the value of f_{16}. It is negative here as it is on the side of 1, that is, remote from 6. Thus, it will be seen that only three lines are necessary for a joint of five members in order to determine the stress in the member, 1-6. For joints with a fewer number of members, only one or two lines are necessary. If there are eccentric connections at the joint, the eccentric moment, *M*, can be taken care of by displacing the force, *R*, a distance equal to $\frac{M}{R}$.

By a similar process the stress in any member may be found, if the stresses in all the other members meeting at the joint are known. The contour, formed by the force, *R*, and the equilibrium lines of the different members, will be known as the "Equilibrium Contour".

To cultivate speed and accuracy in drawing the force diagram for the equilibrium lines, the following method will be found useful. For a truss as shown, Fig. 44, first draw a line perpendicular to the member, 1-2, and measure $ha = S_{12}$. Then, from *a*, draw a line perpendicular to the member, 2-3, and

measure off $ab = S_{23}$. From b, draw a line perpendicular to the member, 2-5, and measure off $bc = S_{25}$. Continue for all the web members of the truss, until the points, e and f, are obtained. Only one-half the truss will be compared, as the other half will be symmetrical. From a, draw a line perpendicular to the member, 1-3, and measure off $ag = S_{13}$. The point, g, should be below a, so that the force, ag, will be downward. Similarly, from all the points, b, c, d, and e, draw lines perpendicular to the respective chord members, and measure off the segments, bk, cj, etc., equal to S_{35}, S_{24}, etc. All the values of the equilibrium lines are then constructed. Next, join the ends, l, j, h, g, etc., with the dotted lines shown. These will give the directions of the external forces, R, which are to be transferred to the truss diagram, as shown. Lastly, the rays, hb, hc, etc., are completed. They are the lines which join the left end of the external force, R, to the inner points of the diagram.

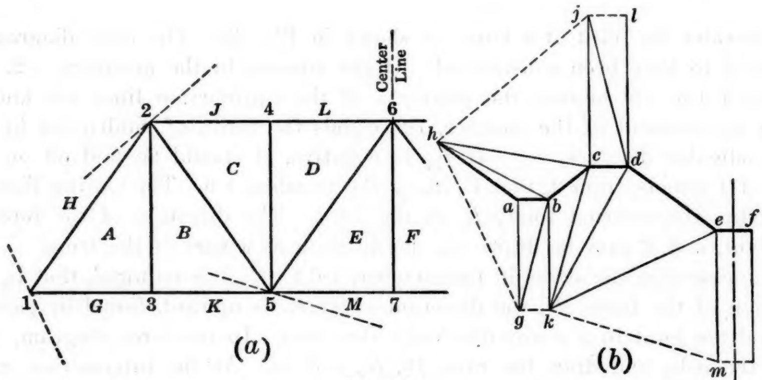


FIG. 44.

The Theory of Deformation Contour

Equation (83) applied to the deformation line of B-5, Fig. 44(a), shows that this deformation line is a graphical representation of the relation between r_{53} and r_{52}. Hence, if r_{53} is known, r_{52} is at once found; but knowing r_{52}, the deformation line, C-5, gives r_{54}, etc. In other words, if r_{53} is known, the deformation lines around Joint (5) permit all the other r's at Joint (5) to be found. This can be expressed analytically, by using Equation (83) in the following manner:

$$r_{52} = \frac{U_{53}}{U_{52}} \cdot r_{53} + \frac{K_{B5}}{U_{52}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (87)$$

$$r_{54} = \frac{U_{52}}{U_{54}} \cdot r_{52} + \frac{K_{C5}}{U_{54}}$$

and substituting from Equation (87):

$$r_{54} = \frac{U_{53}}{U_{54}} \cdot r_{53} + K_{B5} + \frac{K_{C5}}{U_{54}} \ldots\ldots\ldots\ldots\ldots\ldots (88)$$

$$r_{56} = \frac{U_{54}}{U_{56}} \cdot r_{54} + \frac{K_{D5}}{U_{56}}$$

and substituting from Equation (88):

$$r_{56} = \frac{U_{53}}{U_{56}} \cdot r_{53} + K_{B5} + K_{C5} + \frac{K_{D5}}{U_{56}} \qquad (89)$$

$$r_{57} = \frac{U_{56}}{U_{57}} \cdot r_{56} + \frac{K_{E5}}{U_{57}}$$

and substituting from Equation (89):

$$r_{57} = \frac{U_{53}}{U_{57}} \cdot r_{53} + K_{B5} + K_{C5} + K_{D5} + \frac{K_{E5}}{U_{57}} \qquad (90)$$

or,

$$r_{57} = a \cdot r_{53} + b \qquad (91)$$

where a and b are constants; similarly, for r_{52}, r_{54}, and r_{56}.

This is utilized in a simple yet elegant way. In Fig. 45, let 5-a represent r_{53} in magnitude and sign, as determined by earlier construction (assumed) similar to the one to follow.

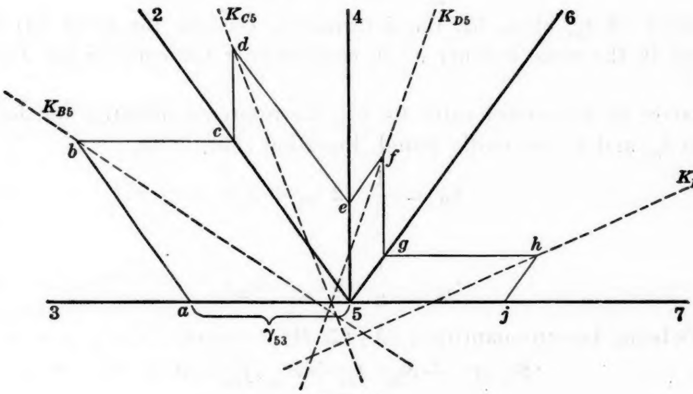


FIG. 45.

Draw ab parallel to the member, 5-2, to cut K_{B5} in b, K_{B5} being the deformation line of the angle, 5, in the triangle, B, and, therefore, supposed to be drawn by methods explained earlier. Draw bc through b parallel to the member, 5-3, to cut the member, 5-2, in c; 5-c is then r_{52}. Draw cd parallel to the member, 5-4, to cut K_{C5} in d, and then draw de parallel to the member, 5-2, to cut the member, 5-4, in e; 5-e is then r_{54}. Repeat this construction to find 5 $j = r_{57}$. This star-shaped diagram is the "deformation contour".

(Note that Dr. Mao introduces two 45° lines toward each other and downward from a and j. The whole diagram enclosed by these and the rays already drawn he calls the "deformation contour". These 45° lines are omitted from this investigation, as the writer fails to see their use; besides, the whole construction in an actual example becomes such a maze of lines that it is advisable to minimize the number.)

In order to use these deformation contours, Mao selects a certain member as the "reference" member for the deformation contour under consideration. It is always the member on the left of the joint. Thus, 5-3, Fig. 44(a), is the

reference member for Joint (5), 2-1 is the reference member for Joint (2), etc.

Suppose, the correct deformation contours for Joints (1) and (2) are known, that is, those contours that give the r's which, when substituted in Equation (85), will yield the desired secondary stresses, f. Substituting r_{12} and r_{21} in Equation (85),

$$f_{12} = \frac{1}{3}\,(2\,r_{12} + r_{21}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (92)$$

Now, because Joint (1) is in equilibrium under $S_{12} \cdot f_{12}$ and $S_{13} \cdot f_{13}$, the two moments at Joint (1),

$$f_{13} = -\frac{S_{12}}{S_{13}} \cdot f_{12} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (93)$$

Therefore, f_{13} and r_{13} are known, the former from Equation (93), and the latter from the assumed deformation contour of Joint (1); but r_{31} is related to f_{13} and r_{13} by Equation (84), therefore,

$$r_{31} = 3\,f_{13} - 2\,r_{13}.$$

By marking off r_{31} along 31, the deformation contour for Joint (3) may be completed in the same manner as in constructing the contour for Joint (5), Fig. 45.

To arrive at the correct value for r_{53}, the reference quantity for Joint (5), note that f_{31} and f_{32} are easily found, Equation (85), from,

$$f_{31} = \frac{1}{3}\,(2\,r_{31} + r_{13})$$

and,

$$f_{32} = \frac{1}{3}\,(2\,r_{32} + r_{23})$$

all the r's being known quantities. As $(\Sigma\, M)_3 = 0$, or,

$$S_{31} \cdot f_{31} + S_{32} \cdot f_{32} + S_{35} \cdot f_{35} = 0$$

or,

$$f_{35} = -\frac{1}{S_{35}}\,(S_{31} \cdot f_{31} + S_{32} \cdot f_{32})$$

The equilibrium polygon of the S's for Joint (3) gives the f_{35} in the quickest way. As r_{35} has already been determined,

$$r_{53} = 3\,f_{35} - 2\,r_{35}$$

from Equation (84), enabling the drawing of the deformation contour for Joint (5).

It is important to note that all this is possible because the relation between any r's and f's, or combinations of r's and f's, is a linear one.

Method of Successive Deduction

With regard to Mao's "Method of Successive Deduction", of which the writer has not carried out an actual example—this method depends on how closely a draftsman can guess the correct values of f_{12} and f_{21}. Suppose two values have been chosen. The foregoing analysis indicates the method of proceeding from joint to joint and fitting in all the deformation contours. When

the last two joints are reached, it will be found that the two relations, $\Sigma M = 0$, for these have not been used at all. After computing the values of ΣM (that is, $\Sigma S . f$), for these joints, making use of the r's found, in general, the resulting ΣM's will differ from zero. Now, assume two new values of f_{12} and f_{21}, repeat the preceding work and obtain a fresh set of values for the ΣM's of the last two joints; if these are nearer zero than the two from the first process, a judicious choice of the f_{12} and f_{21} will give a third approximation of ΣM that will be closer to zero than either of the former two solutions. This process may be continued until the ΣM's for the last two joints are as close to zero as desired. The final set of deformation contours will then give the required secondary stresses. It may be stated here that the element of chance, as in Ritter's method, is not attractive, when there are straightforward methods available.

"Base" and "Vertex" Lines

In order to grasp the idea of Mao's "base" and "vertex" lines, it should be remembered that, by drawing the deformation contours for the r's and the equilibrium polygons for the f's, as already explained, linear relations are established between any desired set of r's, or f's, or combinations of them.

Consider the triangle, B, Fig. 44(a). Mao defines the member, 3-5, as its base (that member of the triangle, that lies in the boundary of the truss). Joint (2) is then its vertex; but Joint (2) is also a joint the reference contour of which is r_{21}. The base will carry the two contours, r_{35} and r_{53}.

Now,

$$r_{53} = a \cdot r_{21} + b \cdot r_{35} + c \qquad (94)$$

in which, a, b, and c are constants.

A proof of this is simple. Suppose, r_{21} and r_{35} to be known; r_{21} establishes a deformation contour round Joint (2), and r_{35} does the same for Joint (3), for, of course, the deformation contour round a joint can be done in either a clockwise or a counterclockwise manner. Now r_{23} and r_{32}, from these contours, give f_{32}, linear in r_{23} and r_{32} and, therefore, linear in r_{21} and r_{35}. Here one must take for granted something that will be proved later, namely, that r_{13} is found from r_{21} and r_{31}, also through a linear relation. Assuming this, as r_{31} and r_{35} are linearly connected, r_{13} is, therefore, a linear expression involving r_{35} and r_{21}. This then gives f_{31}, which is, in turn, only a linear function of r_{21} and r_{35}. Now, f_{35} is a linear function of f_{31} and f_{32} ($f . S = 0$), and therefore, a linear function of r_{21} and r_{35}. Finally, $r_{53} = (3 f_{35} - 2 r_{35})$, and is, therefore, linear with respect to r_{21} and r_{35}.

Suppose, now, that a certain constant value, say, k, be assigned to r_{21}. The relationship, then, between r_{53} and r_{35} will be graphically represented by a straight line. Equation (94) becomes:

$$r_{53} = b \cdot r_{35} + (c + ak) \qquad (95)$$

in which, $(c + a k)$ is a constant. If, now, some value is given to r_{35}, the corresponding value of r_{53} is at once determined.

Lay out these corresponding values along the member, 3-5, as shown in Fig. (46). Through the end points, p and q, draw two lines at 45° to meet in P.

$$x = y + r_{35}$$

and,

$$y = \frac{1}{2}(d - r_{35} - r_{53}) \quad \text{.......................}(96)$$

These relations are evident from a study of Fig. 46. Eliminating r_{35} and r_{53} from Equations (95) and (96),

$$y = \frac{(b+1)}{(b-1)} \,.\, x + \frac{(a\,k + c - d)}{(b-1)} \quad \text{..............}(97)$$

This proves that, for a constant value of r_{21}, P lies on some definite straight line. If r_{21} be given the constant value of zero, the locus of P is called the "base line".

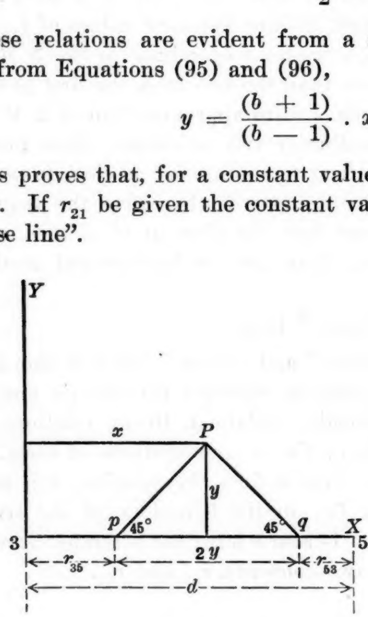


FIG. 46.

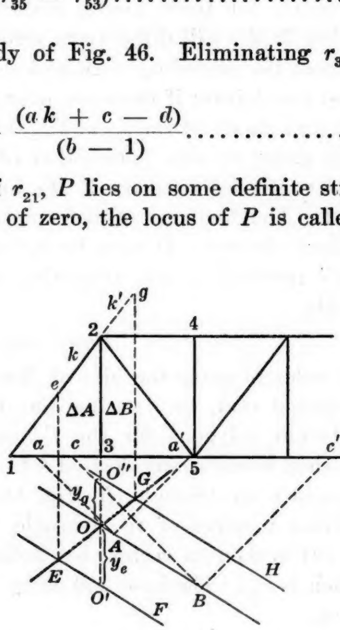


FIG. 47.

It is at once evident from Equation (97) that every value of r_{21} defines a certain straight line, but as this value of r_{21} is incorporated in the constant term of Equation (97) all these lines are parallel to the "base line".

Let $O\,A\,B$ (Fig. 47) be the base line of ΔB. In order to obtain this, r_{21} is assumed to be zero, while two sets of corresponding values of r_{35} and r_{53} are determined. The graphical method of doing this will be shown later.

Let two such sets be ($3a$, $5a'$) and ($3c$, $5c'$). Lines at 45° through a and a' give A, and through c and c' determine B. Let the point of intersection of AB and 2-3 be O. Let $E\,F$ be the line that gives the relationship between r_{35} and r_{53} for a value of, say, $r_{21} = 2e = k$, and let $G\,H$ be that when, say, $r_{21} = 2g = k'$. Draw the perpendiculars, $e\,E$, $g\,G$ to the member, 3-5, and let these cut $E\,F$ and $G\,H$ in E and G, respectively. Join $E\,O$ and $O\,G$. The horizontal distances from E and G to the member, 2-3, are proportional to k and k'. Now, 3-O, 3-O', and 3-O'' are the y-intercepts for the lines, $A\,B$, $E\,F$, and $G.\,H$, respectively, and are, therefore, from Equation (97), respectively, equal to:

$$\frac{c-d}{b-1},\ \frac{a\,k + c - d}{b-1}, \text{ and } \frac{a\,k' + c - d}{b-1}.$$

Hence,

$$O\,O' = \frac{a\,k + c - d}{b-1} - \frac{c-d}{b-1} = \frac{a\,k}{b-1},$$

and,

$$O\,O'' = \frac{a\,k'}{b-1}.$$

The algebraic signs are correct. Therefore, the ratio of $O\,O'$ to $O\,O''$ is $\frac{k}{k'}$.

Further, as the angles, $O\ O'\ E$ and $O\ O''\ G$, are equal, because $G\ H$ and $E\ F$ are parallel, the two triangles, $O\ O'\ E$ and $O\ O''G$, are similar. The angles, $E\ O\ O'$ and $G\ O\ O''$, are, therefore, equal, and, hence, E, O, and G are collinear.

If any other value for r_{21} be chosen, the perpendicular from the end point to the "base", 3-5, will cut the corresponding line of the parallel family in a point on the straight line, $E\ G$. This is Dr. Mao's "vertex line".

(Note that it is now possible to verify the assumption used in the proof for Equation (94). The base and vertex lines of $\Delta\ A$ may be obtained in exactly the same way as for $\Delta\ B$. This means that the relationship between r_{21}, r_{13}, and r_{31} is linear, just as that for r_{21}, r_{35}, and r_{53}, the quantity, r_{21}, being the reference contour in each case. This justifies the statement that f_{31} is a linear function in r_{21} and r_{35}, etc.)

Construction of Base and Vertex Lines

As the base line is a straight line, two points will determine it completely. A splendid check, however, can be introduced here, for if three points are found independently of each other, and are collinear, there is little chance of the base line being wrong. Hence, throughout the work of locating base lines, three points are to be determined for every line.

In determining a vertex line, its accompanying base line having already been found, note that one point on the vertex line is the point of intersection of the base line and the perpendicular from the vertex (see Fig. 47). It remains, therefore, to find two more points, which must be collinear with that on the base line.

Let the base and vertex lines for a truss, shown in Fig. 44(a), be required, This is a Warren truss with verticals, but the method is applicable to all kinds of trusses, the only difference being found in the order of procedure, which is governed by the number of members meeting at the joints. The base of a triangle is always chosen as the member that forms the outline of the truss.

Base and Vertex Lines for $\Delta\ A$.—Either the member, 1-2, or the member, 1-3, may be chosen as the base, as each is part of the outline of the truss. Although only one set of base and vertex lines is necessary for the solution of the problem, it offers a neat check to introduce both sets; this will be explained in the example.

Member 1-3 as Base.—As the vertex of $\Delta\ A$ is Joint (2), the base line is obtained by assuming r_{21} equal to zero, as the member, 2-1, is the reference member for Joint (2).

Some arbitrary value is now assigned to r_{13}, and the deformation contour for Joint (1) completed. This gives the value of r_{12}. As $2\,r_{12} + r_{21} = 3\,f_{12}$ (see Equation (85)), and $r_{21} = 0$ here, let the force, S_{12}, act perpendicularly to

the member, 1-2, at a distance, $2\,r_{12}$, from it. Through the point of intersection of S_{12} and R_1 (the line of action of which has already been determined from the force polygon), draw a line perpendicular to the member, 1-3; the distance from Joint (1) along the member, 1-3, thus found gives the value $3\,f_{13}$; but $r_{31} = 3\,f_{13} - 2\,r_{13}$ (see Equation (84)). Then r_{31} is laid off from Joint (3) along the member, 3-1, care being taken that, if it is negative, it must be measured along the member, 1-3, produced. A line, at 45° to the member, 1-3, and downward to the right, is now drawn through the end of r_{13}, and a line, at 45° to the member, 1-3, downward but toward the left, is drawn through the end of r_{31}; the point of intersection is one of the points on the required base line.

The process is twice repeated, that is, r_{21} remains zero while two more arbitrary values are given to r_{13}. This gives two more points for the base line. If care has been taken with regard to the signs, and the draftsmanship has been accurate, these three points will be found to be collinear, determining the base line for $\Delta\,A$ with the member, 1-3, as base.

By producing the member, 2-3, to cut the base line just found, the first of the three points for the required vertex line is obtained. To get a second point, assume a fairly large value for r_{21} (no longer zero, as for the case of the base line), and some arbitrary value for r_{13}. Using the latter, construct the deformation contour for Joint (1), to find r_{12}. As $3\,f_{12} = 2\,r_{12} + r_{21}$, it is possible to complete the equilibrium polygon round Joint (1), giving $3\,f_{13}$. Now $r_{31} = 3\,f_{13} - 2\,r_{13}$. The two corresponding values of r_{13} and r_{31}, therefore, are known, and by drawing the two 45° lines downward toward each other through the ends of r_{13} and r_{31}, their point of intersection is a point on the straight line parallel to the base line already found. Draw this parallel line; where it cuts the vertical through the end of r_{21}, is one of the points on the required vertex line. Repeat this process with a different r_{21}, fairly large and preferably with a sign different from that already used, this greatly improves the accuracy. The three points should be collinear, giving the required vertex line for $\Delta\,A$ with the member, 1-3, as base.

(Note that Dr. Mao draws his equilibrium polygons with the offsets equal to the values of the f's. It will be noticed that in changing from r's to f's and *vice versa* (see Equation (85)), it is most convenient to work with three times the f-values. An equilibrium polygon around a joint is magnified, therefore, just three times, and not only is time saved, but great accuracy is obtained. Throughout this investigation, the equilibrium polygons are to be understood as being drawn with the $3\,f$-values.)

Member 1-2 as Base.—The vertex is now Joint (3) which has the member, 3-1, for reference member. To obtain the base line, assume r_{31} to be zero, and r_{12} as some arbitrary value. With this r_{12} the deformation contour for Joint (1) is constructed. Now $3\,f_{13} = 2\,r_{13} + r_{31}$, and as $r_{31} = 0$, the equilibrium polygon for Joint (1) is completed by starting with S_{13}, acting at the end of $3\,f_{13} = 2\,r_{13}$. This construction then gives $3\,f_{12}$, but as r_{12} was assumed at the start, the corresponding r_{21} results from $r_{21} = 3\,f_{12} - 2\,r_{12}$. Drawing the 45° lines (to the member, 1-2, in this case) through the ends of r_{12} and r_{21}, deter-

mines the first point on the base line of ΔA, Vertex (3). This process, repeated twice for two new and arbitrary values of r_{12}, gives the two additional points. If the three points are found to be collinear, it is safe to say that the base line has been determined.

To determine the vertex line, assume a fairly large value of r_{31} and use it in conjunction with an arbitrarily chosen r_{12} to obtain the corresponding r_{21} eventually. The difference between this and the analysis for the base line is that the r_{31} in $3 f_{13} = 2 r_{13} + r_{31}$ is no longer zero. The two 45° lines through the ends of r_{12} and r_{21} now determine the point through which a line is drawn parallel to the base line already found. The perpendicular from the end of r_{31} to the base, 1-2, intersects this parallel in a point on the required vertex line. The point where the perpendicular from the Vertex (3) cuts the base line is, of course, another point on the vertex line. A conclusive check is to take another fairly large value of r_{31} and proceed to find a third point, and if this is collinear with the two already found, the vertex line is located.

Base and Vertex Lines for ΔB.—The base for ΔB is the member, 3-5, and the vertex, Joint (2), so that the base line is obtained by making $r_{21} = 0$, the deformation contour for which is drawn. As before, assume a value for r_{35} and find the corresponding value for r_{53}; these two values will determine a point on the base line. On account of the greater number of members entering Joint (3), the process is a little more complicated than for ΔA, but the principle is the same.

For the assumed value of r_{35}, construct a deformation contour for Joint (3), giving r_{32} and r_{31}. Now, as $r_{21} = 0$, the value of r_{13} is found by drawing a 45° line down to the left from the end of r_{31} to the base line of ΔA, Vertex (2), and then a 45° line from this point of intersection to cut the member, 1-3, giving r_{13}. Of all the r's connected with the members entering Joint (3), r_{53} is now the only one unknown. To determine it, the equilibrium polygon for Joint (3) is drawn by making S_{31} act perpendicularly to the member, 3-1, at a distance $3 f_{31} = 2 r_{31} + r_{13}$ from Joint (3) along the member, 3-1, and by making S_{32} act perpendicularly to the member, 3-2, at a distance $3 f_{32} = 2 r_{32} + r_{23}$ from Joint (3) along the member, 3-2; these, together with the resultant R_3, determine the line of action of S_{35}, giving $3 f_{35} = 2 r_{35} + r_{53}$. Hence, $r_{53} = 3 f_{35} - 2 r_{35}$, and the point on the base line of ΔB is found. Two more points are determined, and if the three are collinear the construction for the base line is correct.

The vertex line is determined by assuming fairly large values for r_{21}. These values give certain deformation contours for Joint (2), while they also belong to definite lines parallel to the base line of ΔA, Vertex (2). These parallel lines are found by drawing perpendiculars from the ends of r_{21} to cut the vertex line of ΔA, the parallel lines passing through the points just found. It is these parallel lines that must now be utilized to obtain the values of r_{13} from those of r_{31}, which have themselves come from arbitrary values of r_{35}. The equilibrium polygon for Joint (3) follows, and, hence, r_{53} is determined. The points, determined from corresponding values of r_{35} and r_{53}, are on lines parallel to the base line of ΔB. The points where these parallels cut the verticals from the respective r_{21}'s are on the vertex line required. If two such

points and the point where the perpendicular from Vertex (2) to the base, 3-5, cuts the base line of $\varDelta B$, are collinear, the vertex line for $\varDelta B$ is determined.

Base and Vertex Lines for $\varDelta C$.—The base for $\varDelta C$ is the member, 2-4, and the vertex, Joint (5). The base line, therefore, is obtained by making $r_{53} = 0$ (5-3 being the reference member) and assuming three different values for r_{24}, from which three corresponding values of r_{42} are obtained. In each of these cases r_{53} is zero.

To arrive at a value of r_{42} from a given r_{24} and $r_{53} = 0$, follow the same principle as before. A glance at Fig. 44(*a*) will show that, in order to get to r_{42}, $3\,f_{25}$, $3\,f_{23}$, and $3\,f_{21}$ must first be obtained, with which to draw an equilibrium polygon for Joint (2) that will give $3\,f_{24}$; then only is $r_{24} = 3\,f_{24} - 2\,r_{24}$ obtained. The deformation contour around Joint (2) for the r_{24} under consideration, and the deformation contour around Joint (3) for $r_{53} = 0$, give r_{25} and r_{52}, respectively. Then, $3\,f_{25} = 2\,r_{25} + r_{52}$. The value of r_{21}, from the former contour, also determines the parallels to the base lines of $\varDelta A$ and $\varDelta B$, 2-1 being the reference member for both. A 45° line, from Joint (5) to the parallel to the base line of $\varDelta B$, gives the point from which to draw upward at 45° to obtain r_{35}; this is because $r_{53} = 0$. Use this value of r_{35} for a deformation contour for Joint (3), and with the r_{31} thus found and the parallel to the base line of $\varDelta A$, obtain r_{13}. This gives the deformation contour around Joint (1). The quantities, $3\,f_{25} = 2\,r_{25} + r_{52}$, $3\,f_{23} = 2\,r_{23} + r_{32}$, and $3\,f_{21} = 2\,r_{21} + r_{12}$, are now known, and the problem is solved.

The vertex line is obtained by assuming for r_{53} two values of different magnitude, preferably of different sign. These values give two deformation contours around Joint (5). By giving an arbitrary value to r_{24}, it is possible to proceed exactly as before to get the corresponding r_{42}, the only point to notice being that r_{52} is not zero now, so that the 45° lines must start, not from Joint (5), but from the end of r_{53}. The third point on the vertex line is where the perpendicular from Vertex (5) to the base, 2-4, cuts the base line of $\varDelta C$.

(Note that all the 45° lines are drawn away from the truss, so that the base and vertex lines for $\varDelta C$, $\varDelta D$, $\varDelta G$, and $\varDelta H$ will be situated somewhere above the truss diagram, while those of $\varDelta A$, with Joint (3) as its vertex, lie to the left, those of $\varDelta J$, with Joint (11) as vertex, to the right, and the remainder below the truss diagram. This is done to keep the drawing as clear as possible.)

Solution of Problem by Deformation Contour

The solution depends on the determination of all the base and vertex lines of all the triangles. The foregoing explanation appears to be difficult because it is long; this seems to be unavoidable. The writer found that, by merely reading Dr. Mao's thesis, his method seemed very difficult, but after going through the computations of an actual example the whole principle became exceedingly clear and simple. Any one who is eager to know Mao's work thoroughly is strongly advised, therefore, to work out a simple example; if time will not permit, a neat freehand sketch, accompanying the reading, will serve the purpose.

The base and vertex lines for the triangles, Fig. 44(*a*), are found in the following order, the vertex line of a triangle being determined immediately after the base line of the triangle in question has been determined: ΔA with either Joint (2) or Joint (3) as vertex, the order of procedure being immaterial; ΔB with Joint (2) as vertex; ΔC with Joint (5) as vertex; ΔD with Joint (5) as vertex, etc.

When this stage is reached, it becomes evident that ΔJ has the member, 10-12, in the outline of the truss, but instead of obtaining only one set of base and vertex lines with the vertex at Joint (11), two are obtained, one by considering the equilibrium of Joint (12), and the second that of Joint (10). It is on this property that the solution of the whole problem depends. It is the same idea that underlies the method of Müller-Breslau and that of Mohr (by elastic weights), where the two last joints give two equations wherewith to solve the two values assumed in the beginning.

Assume that the two sets of base and vertex lines of ΔJ, Vertex (11), are drawn and, of course, all the preceding ones. The reference member for ΔJ, Vertex (11), is 11-9. Assume any arbitrary value for $r_{11\cdot 9}$. A perpendicular, from the end point of $r_{11\cdot 9}$ to the member, 12-10, will cut the two vertex lines of the respective sets for ΔJ, Vertex (11), in two points, through which parallels to the respective base lines are then drawn. These parallels determine a point, most important inasmuch as it is common to the two sets. Proceed to draw 45° lines to the member, 10-12, from this point. Two values are determined, namely, $r_{10\cdot 12}$ and $r_{12\cdot 10}$. Complete the deformation contours round Joints (10) and (12). The value of $r_{10\cdot 8}$ found will enable the determination of a parallel to the base line of ΔJ, Vertex 10 (10-8 is the reference member in this case); this parallel must be used in conjunction with $r_{12\cdot 11}$, from Contour (12), to obtain $r_{11\cdot 12}$. Complete the deformation contour for Joint (11). The value of $r_{11\cdot 9}$ thus found will, in general, be quite different from that assumed at the start. The object is to obtain an exact coincidence. A second value of $r_{11\cdot 9}$ is judiciously assumed; it is advisable that this value approximate the one found in the process just described. The above procedure is then repeated. It will be found that the resultant value of $r_{11\cdot 9}$ will come fairly close to that assumed. This process of choosing $r_{11\cdot 9}$ and drawing is repeated a few times; after the fourth attempt, it is customary to check the value of $r_{11\cdot 9}$ last chosen.

Assume that the correct $r_{11\cdot 9}$, with the attendent deformation contours round Joints (12), (11), and (10), are now known. Draw the perpendicular from the end of $r_{10\cdot 8}$ to cut the vertex line of ΔI; through this point, draw a line parallel to the base line of ΔI. Use this parallel to determine $r_{9\cdot 11}$ from $r_{11\cdot 9}$. Complete the deformation contour of Joint (9) giving $r_{9\cdot 7}$. This is now the reference contour for ΔH and ΔG. Using it, determine $r_{8\cdot 10}$ from $r_{10\cdot 8}$, and complete the deformation contour for Joint (8). This process is continued for all the joints. ΔA will have been reached and all the deformation contours of all the joints will have been drawn, without having made use of the base and vertex lines of ΔA, Vertex (3). Their usefulness lies in the following: Draw a perpendicular to the member, 1-2, from the

end of r_{31}, this being the reference contour. This line cuts the vertex line in a point through which a line is now drawn parallel to the base line. Draw lines at 45° to the member, 1-2, from the ends of r_{21} and r_{12}. If the work has been accurate, these 45° lines must intersect each other on the parallel line just drawn.

Consider, for example, Joint (6). The r's of all the members entering Joint (6) have been found. Lay off from Joint (6) the quantities, $(2\ r_{68} + r_{86})$, $(2\ r_{69} + r_{96})$, $(2\ r_{67} + r_{76})$, $(2\ r_{65} + r_{56})$, and $(2\ r_{64} + r_{46})$, along the members, 6-8, 6-9, 6-7, 6-5, and 6-4, respectively. These are now the $3\ f$'s of the members. The resultant, R_6, of the forces, S_{68}, S_{69}, S_{67}, S_{65}, and S_{64}, is known in magnitude and direction from the force polygon, while its point of application is Joint (6). Draw the equilibrium polygon, considering everything known but the point of application of S_{64}. This will give the quantity, $3\ f_{64}$, which, together with the quantities, $3\ f_{68}$, $3\ f_{69}$, $3\ f_{67}$, and $3\ f_{65}$, will cause equilibrium round Joint (6), that is, $(\Sigma\ S\ .\ 3\ f)_6 = 0$. If the work has been exact this value, $3\ f_{64}$, will correspond exactly with $3\ f_{64}$ found from the deformation contours, namely, $2\ r_{64} + r_{46}$; the closeness in agreement will be a measure of the accuracy of the drawing. Repeat this for all the joints. If the agreement is good in all cases, the values of the $3\ f$'s may be considered to be correct. A third of each of these values gives the corresponding secondary stress.

Analytical Solution by the Scientific Arrangement of Computations

Introduction.—This method is simply the application of Equations (81) to (85), together with the equations found at the joints, namely, $\Sigma\ M = 0$. Its underlying principles, therefore, are easy. The difficulty is that, in common with the methods of deformation contour and successive deduction, it needs a long explanation, accompanied by a diagram. The method is indeed very much like that of Müller-Breslau; Mao also assumes two quantities known at the start, and using these throughout the process, finally arrives at two simultaneous equations the solution of which determines the two assumed quantities. The assumed quantities in this method are the secondary stresses of the first member of the truss.

Mao uses two truss diagrams. He has an ingenious device for determining the changes of the angle, $\Delta\ \alpha$, on the first diagram. On the second diagram, he places a number of skeleton tables which must be filled out in a certain definite way; each place in every table has a definite significance and seldom is any quantity repeated.

A point that must be emphasized again is this: If the two assumed secondary stresses of the first member are x and y, with reference to Equations (81) to (85) and the $\Sigma\ M = 0$ equations, that is, $\Sigma\ S\ .\ f = 0$ equations (see Equation (79)), it is to be noted that every relation to be established during the analysis will be one of the first degree in x and y.

Application.—As the theory has been proved, the method will be illustrated by actually applying it to a simple case. Consider a truss composed of two triangles; the solution of this truss embodies all the elements found in one with

a greater number of triangular elements. Further, as most trusses are composed of right-angled triangles, Mao's method for finding the changes of angle will be given for right-angled triangles only. It is really Ritter's method (see page 977).

In Fig. 48 are two diagrams, representing a truss composed of two triangles, A and B. On Fig. 48 (*a*), write out the lengths, L, in feet, moments of inertia, I, in inch units, and the extreme fiber distances, y, in inches, as shown. These are obtained from the dimensions of the truss. Now imagine the truss to be loaded, giving the unit stresses, P, in pounds per square inch, which are also written on the members. Obtain also, from the dimensions, the half values of the co-tangents of α and β.

In each of the triangles, A and B, Fig. 48(*a*) make a table as shown, observing the proper positions for the various values. The quantities at the three ends are K_{A1}, K_{A2}, and K_{A3}. These K's are the half values of the product of the modulus of elasticity and the changes of the angle, $\Delta\,\alpha$. (Equation (81); see, also, "Preliminary Considerations," page 1010.) A comparison between this method, and "The Changes of Angle" as discussed on page 1012, will show that the method is legitimate. K_{A1} is the change of Angle (1) in Triangle (A), and is found at the end of the A-table, nearest to Joint (1). Similarly, for all the other K's.

In Fig. 48(*b*), directly below Joints (1), (3), and (4), and directly above Joint (2), place the $\Sigma\,M$- and the $\Sigma\,K$-tables as shown. In the $\Sigma\,K$-table, include all the members that meet in the joint except the first. This comprises the members, 3-2 and 3-4, for Joint (3). The first member of a joint in the bottom chord is that which is first met on entering the truss in a clockwise direction round the joint; the first member of a joint in the top chord is that which is first met on entering the truss in a counterclockwise direction round the joint.

In the $\Sigma\,M$-tables all the members are included and again the clockwise direction for the bottom chord joints and the counterclockwise direction for the top chord joints are observed.

On the end chord, 1-2, a U-table is constructed, as shown; r_{21} the top letter, refers to the top end of the member, and the bottom letter, r_{12}, to the bottom end. Recalling the method of the deformation contour, these r's are the "contours" of the respective ends of the member.

On each of the two bottom chords, 1-3 and 3-4, a table is constructed, one for U_{13} and S_{13}, the other for U_{34} and S_{34}; these quantities are again obtained from Fig. 48(*a*). Note that the top r and f refer to the left end of the member, while the bottom r and f refer to the right end. For top chord members, tables identical with these U-tables would be used.

On each of the remaining members, Fig. 48(*b*), namely, 2-3 and 2-4, place two tables, as shown. It will be seen that the f and the r at the top refer to the top end of the member, while the lower r and f refer to the lower end of the member. The letters on the U_{24}-tables need no explanation; the remaining tables, namely, one on each end of every member, will be explained

later, as they come into use only after the problem of solving for the x and the y have been completed.

The first columns of the ΣM-tables are now filled with the values of the S for the members denoted at the sides. Thus, insert S_{31}, S_{32}, and S_{34} for the ΣM_3-table.

Consider one of the bottom ΣK-tables, say, the ΣK_3-table. Opposite K_{A3} and below the member, 3-2, place the value of K_{A3}, found from Fig. 48(a). In the space directly below, fill in the value of K_{B3} already found sum the two quantities and place the result in the adjacent space. This value will now refer to the member, 3-4. If there were more members at the joint, the table would be extended in the following manner: Suppose there were a K_{x3} and a member, 3-n: Directly below the value last found, would be placed the value of K_{x3}; add the two and put the sum in the adjacent space, and so on, for more members.

The ΣK-tables for the upper joints differ in that the rotation round the lower joints is clockwise, while that round the upper joints is counterclockwise; hence, use the negative values of the K's in the top K-tables. Otherwise, they are exactly similar to the bottom K-tables.

Recalling the method of deformation contour, the member to the left of a joint is the reference member. Therefore, the set of ΣK-tables is just a neat method of tabulating the changes of angle between the reference member and the other members of a joint.

The next step is to fill in the spaces shown extending out from the main tables, Fig. 48(b). As is evident from Equations (87) to (90), the r of a member at a joint is connected linearly with the reference, r, of that joint, and the constant term in such an equation is the quantity obtained by dividing the sum of the K's of all the angles between the member in question and the reference member by the U of the member considered. Hence, in the space of the U_{13}-table, put the value, $\frac{K_{A1}}{U_{13}}$, and in the space of the U_{34}-table, the value, $\frac{K_3}{U_{34}}$, that is, $\frac{K_{A3} + K_{B3}}{U_{34}}$. In the spaces of the center row of U-tables, however, put the third values, thus, for the lower space in the U_{23}-table, insert $\frac{K_{A3}}{3\ U_{23}}$, and for the upper space, $\frac{-K_{A2}}{3\ U_{23}}$. All the spaces that have been filled up to this stage are marked by crosses.

To proceed with the solution, let the secondary stresses in the member, 1-2, have the values, x and y, that is, let

$$f_{12} = x \qquad (98)$$

and,

$$f_{21} = y \qquad (99)$$

The spaces of the row opposite f_{12} in the U_{12}-table, are filled in with the numbers, $+1, 0, 0$, while those opposite f_{21} are filled in with the number, $0, +1, 0$. These are the coefficients of the x and y and the constant term in the expres-

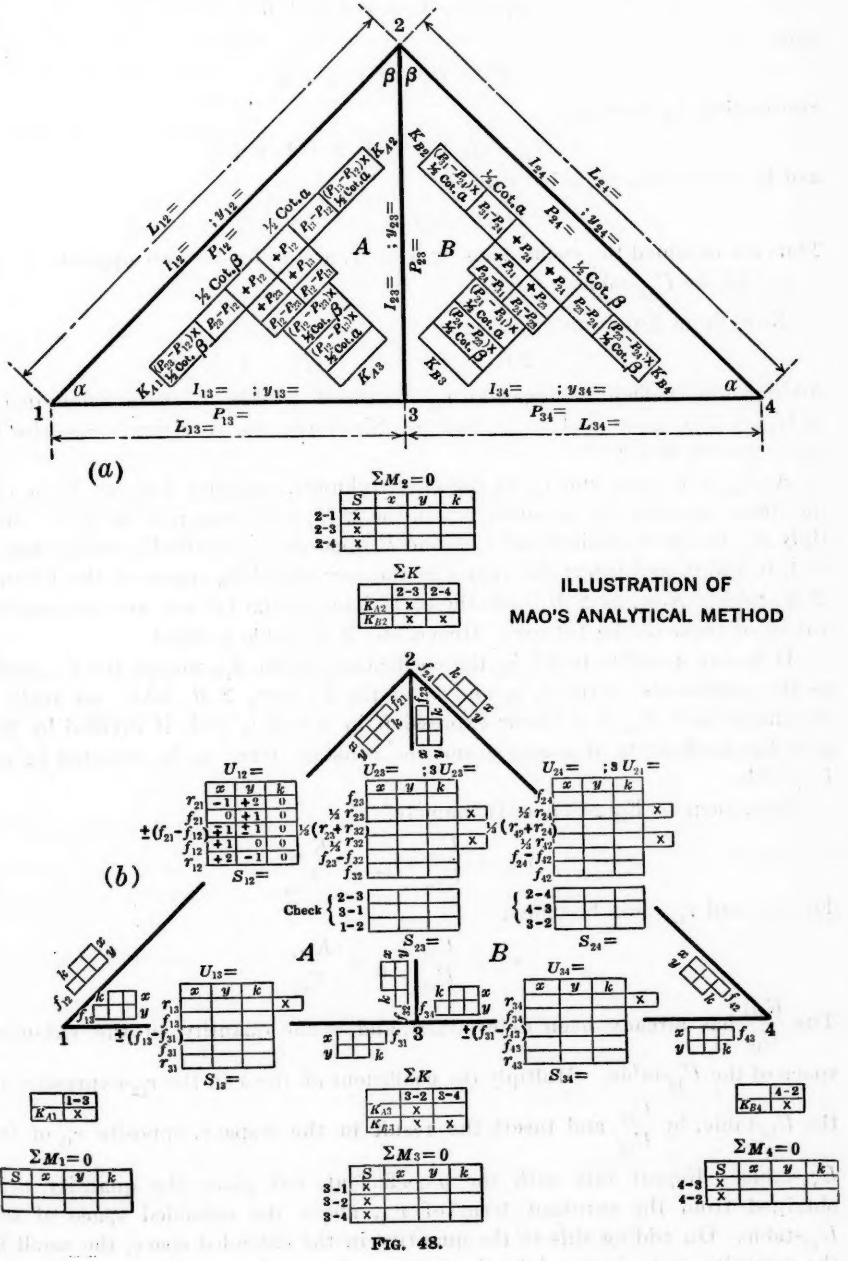

(a)
(b)
ILLUSTRATION OF
MAO'S ANALYTICAL METHOD
$\Sigma M_2 = 0$
ΣK
$\Sigma M_1 = 0$
$\Sigma M_3 = 0$
$\Sigma M_4 = 0$
$U_{12}=$
$S_{12}=$
$U_{23}=$; $3\,U_{23}=$
$S_{23}=$
$U_{24}=$; $3\,U_{24}=$
$S_{24}=$
$U_{13}=$
$S_{13}=$
$U_{34}=$
$S_{34}=$
Check
A
B

FIG. 48.

sions for the f's in terms of x and y, for, evidently, from Equations (98) and (99),

$$f_{12} = +1 \,.\, x + 0 \,.\, y + 0$$

and,

$$f_{21} = 0 \,.\, x + 1 \,.\, y + 0$$

Subtracting f_{12} from f_{21},

$$+ (f_{21} - f_{12}) = -1 \,.\, x + 1 \,.\, y + 0$$

and by taking the negative value,

$$- (f_{21} - f_{12}) = + x - y + 0$$

Thus are obtained the coefficients, ∓ 1, ± 1, and 0, for the row opposite $\pm (f_{21} - f_{12})$ in the U_{12}-table.

Now, from Equation (82),

$$r_{21} = 2 f_{21} - f_{12} = f_{21} + (f_{21} - f_{12}).$$

Adding the coefficients opposite f_{21} in the U_{12}-table to those belonging to $+ (f_{21} - f_{12})$, gives -1, $+2$, and 0. Similarly, the coefficients opposite r_{12} are $+2$, -1, and 0.

As f_{12} is known, and f_{13} is the only unknown quantity left for Joint (1), the spaces opposite the member, 1-2, in the ΣM_1-table may now be filled. Multiply S_{12} by the coefficients of x, y, and k, opposite f_{12} in the U_{12}-table, that is, $+1$, 0, and 0, and insert the values in the corresponding spaces of the 1-2 row, ΣM_1-table. Now, as $\Sigma M_1 = 0$, the coefficients in the 1-3 row are the negative values of those of the 1-2 row. Hence, the ΣM_1-table is filled.

It is now possible to fill in the coefficients in the f_{12} row of the U_{13}-table, as the coefficients of the x, y, and k, of the 1-3 row, ΣM_1-table, are really a statement that M_{13} is a linear expression in x and y, and, if divided by S_{13}, give the coefficients of x and y and the constant term, to be inserted in the U_{13}-table.

Now, turn to Equation (83), namely,

$$r_{lm} = \frac{U_{ln}}{U_{lm}} \,.\, r_{ln} + \frac{K_{mln}}{U_{lm}}$$

For r_{13} and r_{12}, this becomes:

$$r_{13} = \frac{U_{12}}{U_{13}} \,.\, r_{12} + \frac{K_{A1}}{U_{13}}$$

The $\dfrac{K_{A1}}{U_{13}}$ has already been determined, and is the quantity in the extended space of the U_{13}-table. Multiply the coefficient of the x in the r_{12}-expression in the U_{12}-table, by $\dfrac{U_{12}}{U_{13}}$, and insert the result in the x-space, opposite r_{13} of the U_{13}-table. Repeat this with the y-coefficient, but place the quantity thus obtained from the constant term of r_{12}, above the extended space of the U_{13}-table. On adding this to the quantity in the extended space, the result is the quantity to be inserted in the k-space of r_{13}. It must be noted that, as soon as two rows of any U-table are filled in, the remainder follow, from

Equations (98) and (99). Hence, subtracting the f_{13}-coefficients from the r_{13}-coefficients, gives the $+(f_{13}-f_{31})$-coefficients; the correct signs to take care of the $\pm(f_{13}-f_{31})$ are placed on the side. Adding these coefficients, with the lower signs, to the coefficients in the row immediately above, gives the coefficients of x, y and the constant term for f_{31}. Finally, adding the set of quantities in the f_{31}-row to that immediately above, gives r_{31}.

Turning now to the U_{23}-table, note that Equation (83) becomes:

$$r_{32} = \frac{U_{31}}{U_{32}} \cdot r_{31} + \frac{K_{A3}}{U_{32}}$$

or,

$$\frac{1}{3} r_{32} = \frac{U_{31}}{3\,U_{32}} \cdot r_{31} + \frac{K_{A3}}{3\,U_{32}}$$

from which to fill in the spaces for $\frac{1}{3} r_{32}$. The constant term of this expression is already in the extended space opposite $\frac{1}{3} r_{32}$. The factor with which to multiply the various quantities in the r_{31}-row of the U_{13}-table is, therefore, $\frac{U_{31}}{3\,U_{32}}$. Note, here again, the k-term resulting from this operation is placed immediately above the extended space of $\frac{1}{3} r_{32}$. From the fact that,

$$\frac{1}{3} r_{23} = \frac{U_{21}}{3\,U_{23}} \cdot r_{21} + \frac{-K_{A2}}{3\,U_{23}}$$

proceed in exactly a similar way to fill the spaces of the $\frac{1}{3} r_{23}$-row of the U_{23}-table. With two rows completed the remaining spaces are readily filled in. Adding the $\frac{1}{3} r_{23}$-row to the $\frac{1}{3} r_{32}$-row, fills the spaces that lie between these rows. Adding the $\frac{1}{3} r_{23}$-row to the $\frac{1}{3}(r_{23}+r_{32})$-row, gives the f_{23}-row; a reference to Equation (85) proves this. The remainder follows. The U-tables of the three members of the triangle, A, are now complete.

To fill in the check-table on the member, 2-3, turn to Equation (81) and express it in terms of the quantities of the ΔA. Thus,

$$\left.\begin{aligned} K_{A1} &= U_{13}\,(2f_{13}-f_{31}) - U_{12}\,(2f_{12}-f_{21}) \\ K_{A2} &= U_{21}\,(2f_{21}-f_{12}) - U_{23}\,(2f_{23}-f_{32}) \\ K_{A3} &= U_{32}\,(2f_{32}-f_{23}) - U_{31}\,(2f_{31}-f_{13}) \end{aligned}\right\} \ldots\ldots\ldots\ldots (100)$$

However, as the sum of the three angles of a triangle is constant,

$$K_{A1} + K_{A2} + K_{A3} = 0$$

Note that $U_{13} = U_{31}$, $U_{21} = U_{12}$, and $U_{32} = U_{23}$. Adding Equations (100) and dividing by 3,

$$U_{13}\,(f_{13}-f_{31}) + U_{21}\,(f_{21}-f_{12}) + U_{32}\,(f_{32}-f_{23}) = 0$$

that is,

$$-U_{13}\,(f_{13}-f_{31}) - U_{21}\,(f_{21}-f_{12}) + U_{32}\,(f_{23}-f_{32}) = 0.$$

Hence, multiply the $(f_{23} - f_{32})$-coefficients in the U_{23}-table by U_{23}, to obtain the coefficients in the 2-3 row of the check-table. Corresponding quantities are found from the U_{13} and U_{12}-tables, noting that the lower signs of the coefficients are used in each case. If the sum of the x-coefficients in the check-table is zero, of the y-coefficients, and also the sum of the constant terms, the check is complete. In filling all the remaining tables, it will be found that the relation, $\Sigma M = 0$, is not used at all in completing the ΣM-tables for the last two joints in this case, the ΣM_2 and ΣM_4-tables.

Adding the three columns in each, an expression is obtained from the M_2-table of the form $(a x + b y + c)$; but as $\Sigma M_2 = 0$,

$$a x + b y + c = 0$$

Similarly, from $\Sigma M_4 = 0$, results an equation such as,

$$a' x + b' y + c' = 0$$

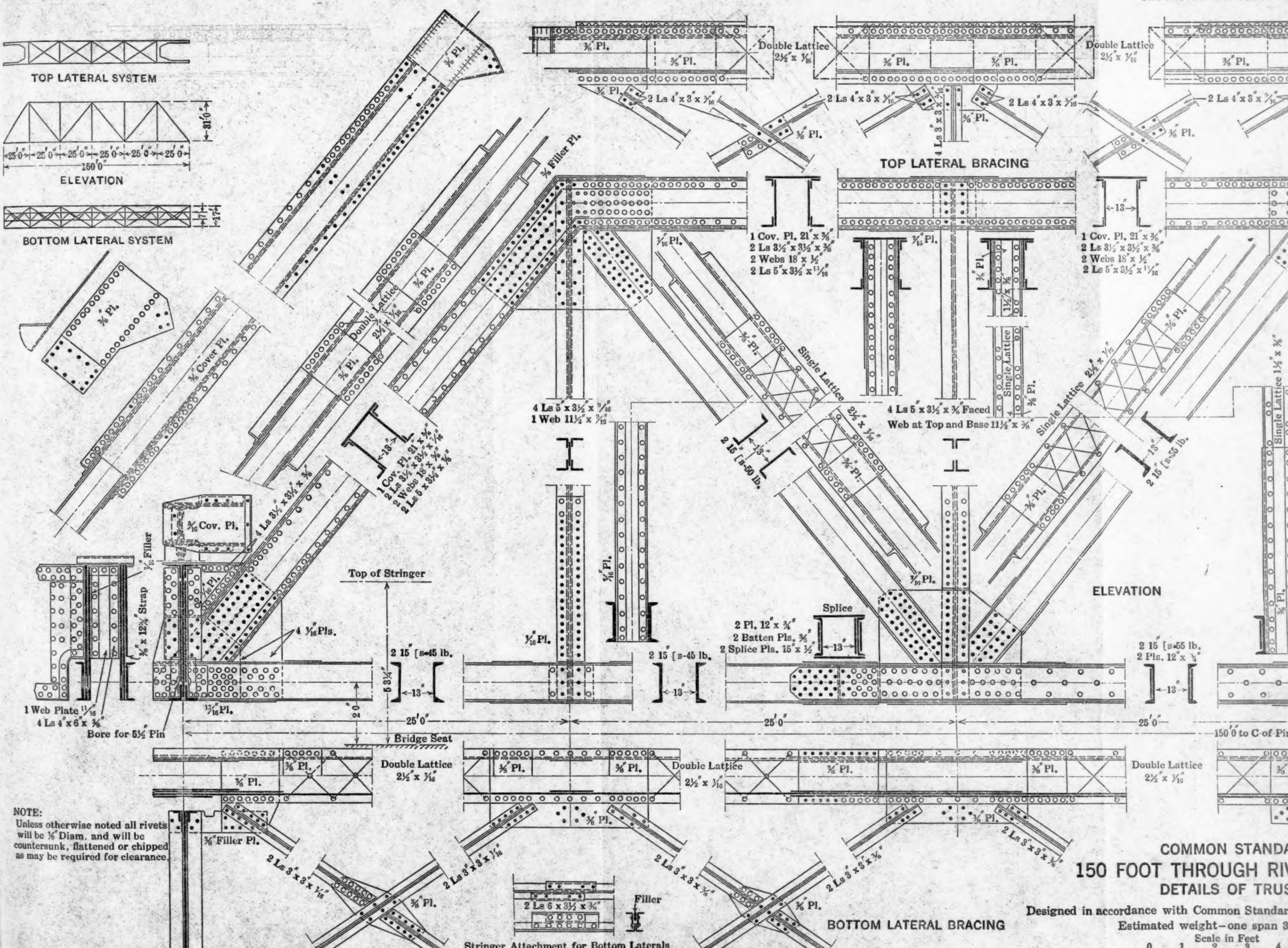
By solving for x and y, from these equations are obtained what were assumed to be known at the outset, namely, f_{12} and f_{21}. Next fill in the small tables around the joints (Fig. 48(b)); these tables have been mentioned, but not explained. Take the table for f_{31}. Transfer the k-value, found in the f_{31}-row of the U_{13}-table, to the space next to k in the f_{31}-table. Multiply the x-coefficient of f_{31} in the U_{13}-table by the value of x just found, and insert the result in the space next to x in the f_{31}-table, and, similarly, for the y-space. Now, add these two quantities, and insert the result in the space above that containing the k-value. Add the k-value to the quantity above it, and place the result in the last space of the f_{31}-table. The quantity thus obtained is the secondary stress of the member, 1-3, for End 3. The computer should proceed in the same manner for all the f-tables and thus determine all the secondary stresses.

PART III.—A CRITICAL COMPARISON BETWEEN THE METHODS OF SOLVING SECONDARY STRESSES IN BRIDGE TRUSSES

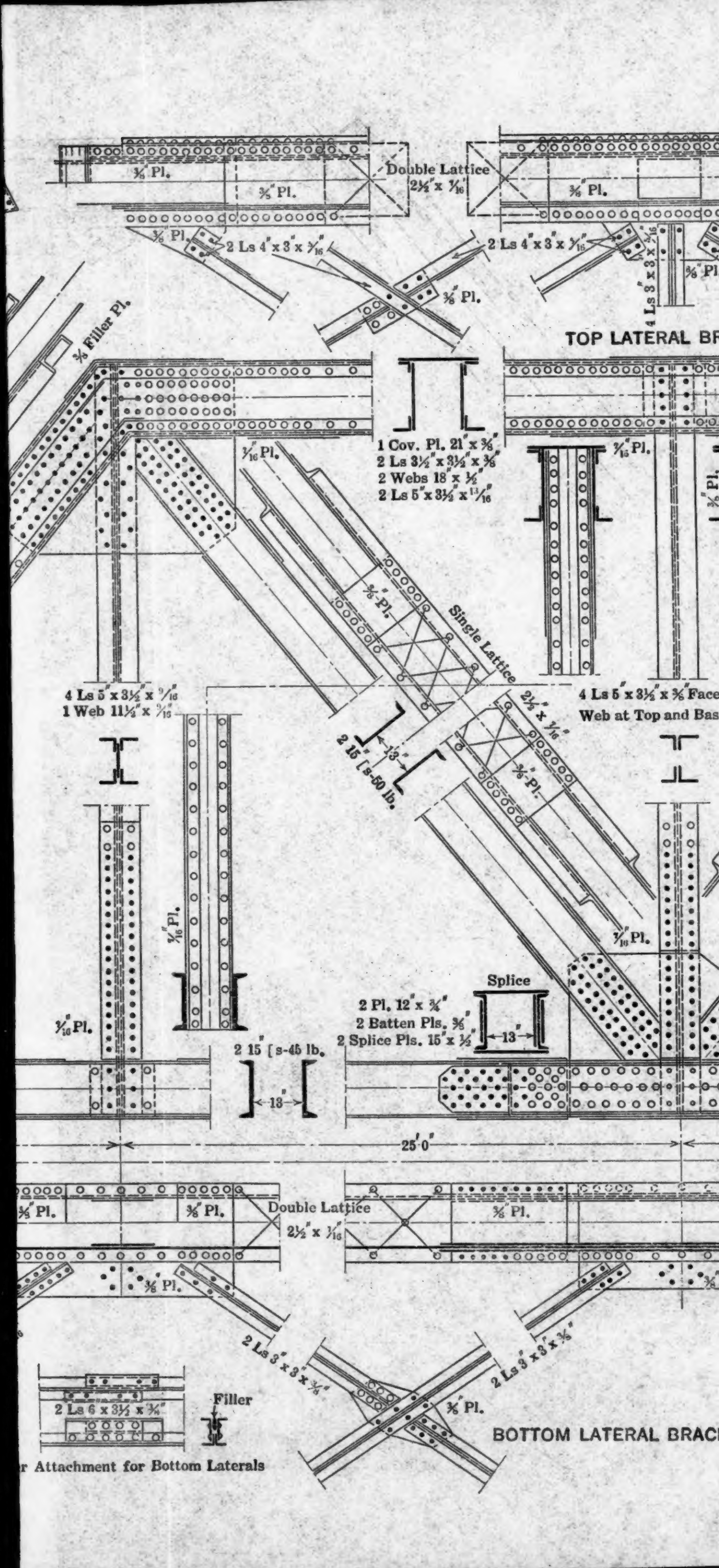
1.—Introduction

Many methods of computing the secondary stresses in the trusses of a bridge have been described. At once the question arises: "Which one is the best in its practical application"? In order to arrive at a definite conclusion, it is proposed to apply each method in turn to the same bridge under a load, identical for all the methods. Many comparisons have been made, generally for a bridge loaded symmetrically. This comparison will be for an unsymmetrical load, by which each method will be tested in the severest manner; the one best enduring the test will undoubtedly be the best method in practice.

Plate VI and Figs. 49 and 50 show the design of a 150-ft., common standard through riveted Warren truss bridge in present use. The trusses are composed entirely of triangular elements and if they had frictionless hinges would be regarded as statically determinate. Therefore, each of the methods may be applied without adding any further explanations to the theoretical investigations already given.

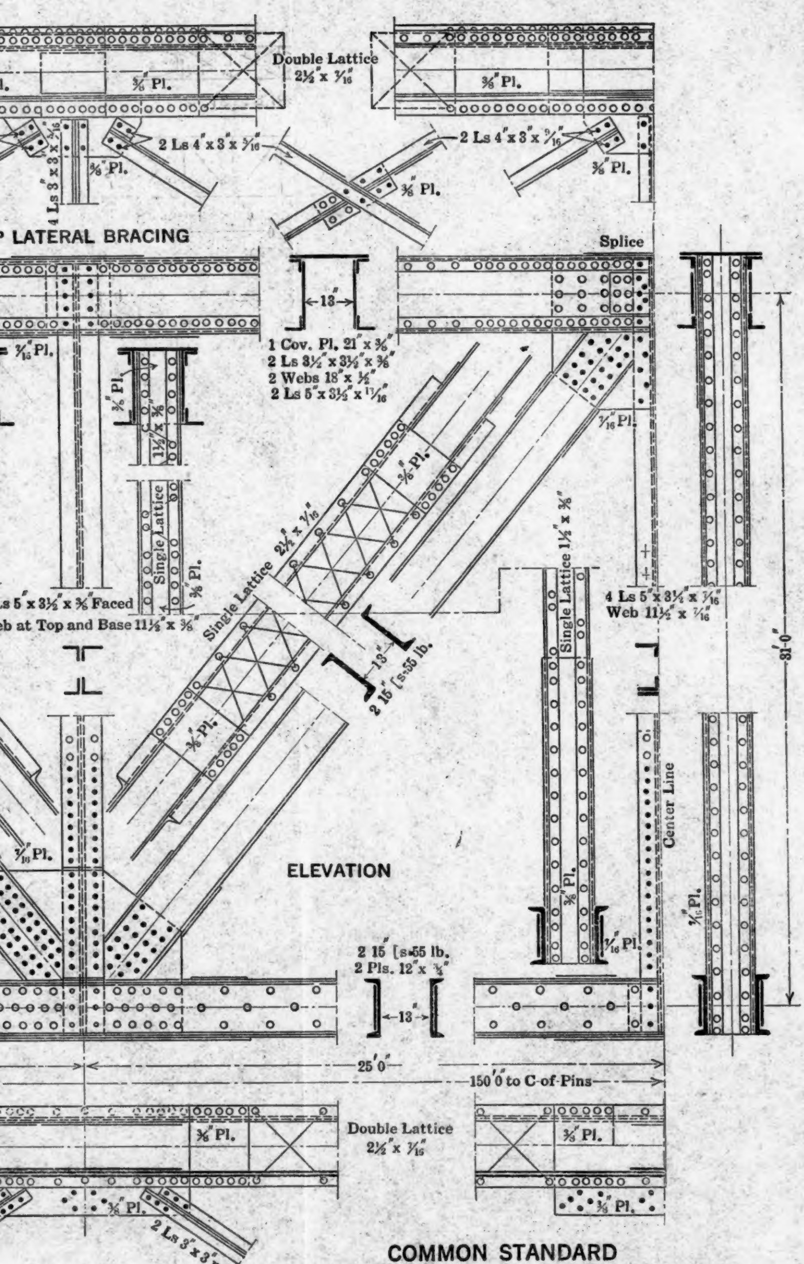


COMMON STANDAR
150 FOOT THROUGH RIVE
DETAILS OF TRUSSE
Designed in accordance with Common Standard S
Estimated weight—one span 311



Double Lattice
2½" x 5/16"
3/8" Pl.
2 Ls 4" x 3" x 5/16"
3/8 Filler Pl.
4 Ls 3 x 3 x 5/16
TOP LATERAL BR
1 Cov. Pl. 21" x 3/8"
2 Ls 3½" x 3½" x 3/8"
2 Webs 18" x ½"
2 Ls 5" x 3½" x 11/16"
5/16" Pl.
Single Lattice
2½" x 5/16"
4 Ls 5" x 3½" x 5/16"
1 Web 11½" x 5/16"
2 15" [s-50 lb.
13"
4 Ls 5" x 3½" x 3/8" Face
Web at Top and Bas
Splice
2 Pl. 12" x ¾"
2 Batten Pls. 3/8"
2 Splice Pls. 15" x ½"
2 15" [s-45 lb.
25' 0"
Double Lattice
2½" x 5/16"
2 Ls 3" x 3" x 3/8"
2 Ls 6 x 3½ x ¾"
Filler
r Attachment for Bottom Laterals
BOTTOM LATERAL BRAC

PLATE VI.
PAPERS, AM. SOC. C. E.
SEPTEMBER, 1924
VON ABO ON
SECONDARY STRESSES IN BRIDGES.



COMMON STANDARD

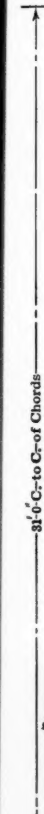
150 FOOT THROUGH RIVETED SPAN

DETAILS OF TRUSSES

Designed in accordance with Common Standard Specifications No. 1006.
Estimated weight—one span 311 000 lb.

Scale in Feet
0 1 2 3 4 5

Pa



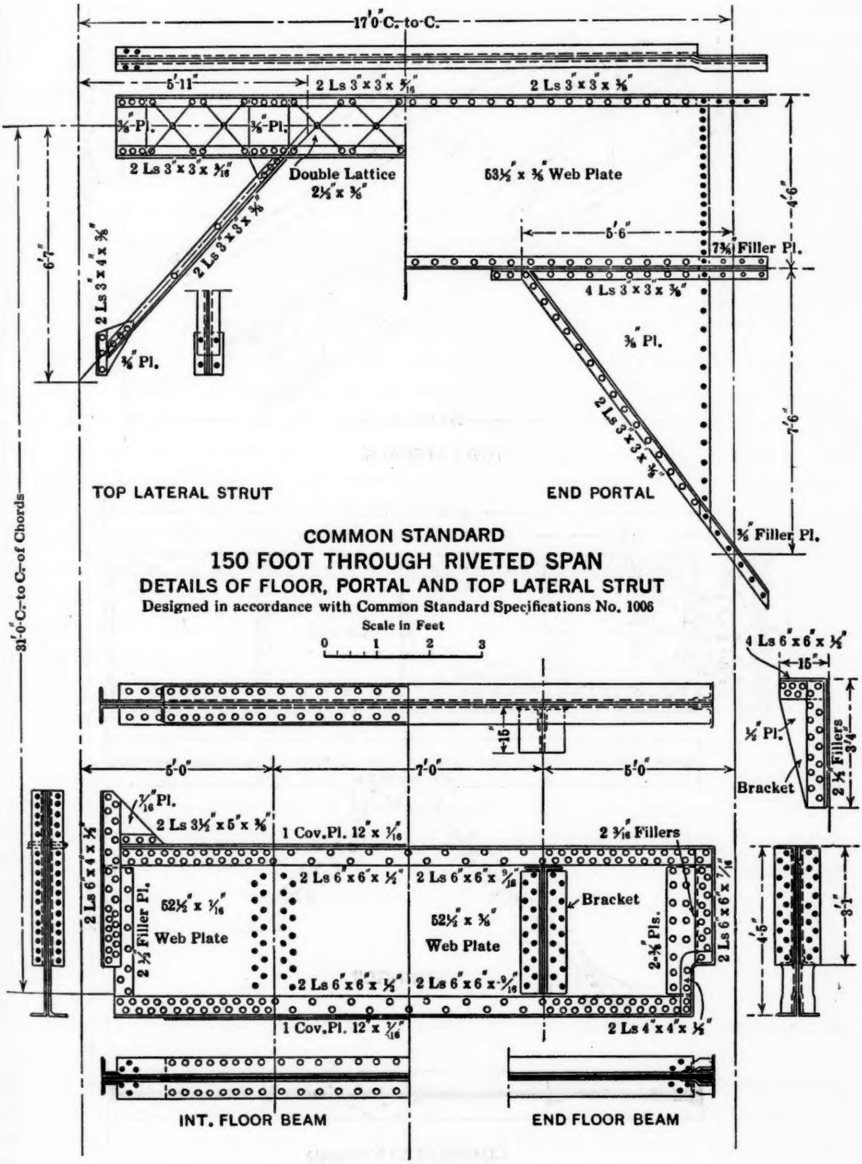
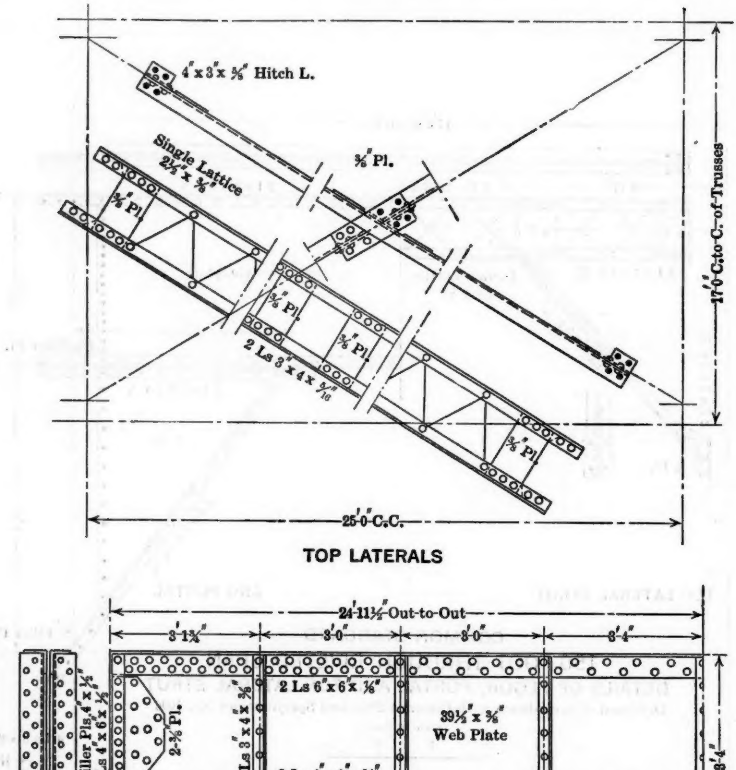

17'0" C. to C.
5'11"
2 Ls 3"x3"x 5/16"
2 Ls 3"x3"x 3/8"
3/8" Pl.
3/8" Pl.
2 Ls 3"x3"x 5/16"
Double Lattice 2½"x 3/8"
53½" x 3/8" Web Plate
4'6"
6'7"
2 Ls 3"x4"x 3/8"
2 Ls 3"x3"x 3/8"
3/8" Pl.
5'6"
7¾" Filler Pl.
4 Ls 3"x3"x 3/8"
3/8" Pl.
2 Ls 3"x3"x 3/8"
7'6"
3/8" Filler Pl.
31'0" C. to C. of Chords
TOP LATERAL STRUT
END PORTAL
COMMON STANDARD
150 FOOT THROUGH RIVETED SPAN
DETAILS OF FLOOR, PORTAL AND TOP LATERAL STRUT
Designed in accordance with Common Standard Specifications No. 1006
Scale in Feet
0 1 2 3
4 Ls 6"x6"x ½"
15"
½" Pl.
2 ½" Fillers
3'4"
Bracket
1'5"
5'0"
7'0"
5'0"
5/16" Pl.
2 Ls 3½"x5"x 3/8"
1 Cov. Pl. 12"x 7/16"
2 5/16" Fillers
2 Ls 6"x4"x ½"
2 ½" Filler Pl.
52½" x 7/16" Web Plate
2 Ls 6"x6"x ½"
2 Ls 6"x6"x 9/16"
52½" x 3/8" Web Plate
Bracket
2-3/8" Pls.
2 Ls 6"x6"x 7/16"
4'5"
3'1"
2 Ls 6"x6"x ½"
2 Ls 6"x6"x 9/16"
1 Cov. Pl. 12"x 7/16"
2 Ls 4"x4"x ½"
INT. FLOOR BEAM
END FLOOR BEAM

FIG. 49.



COMMON STANDARD

150 FOOT THROUGH RIVETED SPAN

DETAILS OF FLOOR AND LATERALS

Designed in accordance with
Common Standard Specifications No. 1006

Scale in Feet
0 1 2 3

FIG. 50.

Because influence lines are to be used extensively in this study, a single load of definite value is chosen. Influence lines deal with units of load; such a unit may be 1 lb., 1 kip, or 1 000 lb., 1 ton, etc. Here the kip will be adopted as the unit. When the truss has a load of 1 kip at the central panel point of the bottom chord, the left-hand reaction is ½ kip. If the load were 1½ kips placed at the first panel point to the right of the center, the left-hand reaction would again be ½ kip. Finally, if it were 3 kips placed at the panel point to the right, two panel distances from the center, the left-hand reaction would still be ½ kip. It will be found that several members on the left side of the truss are stressed equally under the 1-kip and 1½-kip loads, while an even greater number are stressed equally under the 1½-kip and the 3-kip loads. This method of loading a bridge for the determination of its secondary stresses is due to Professor Turneaure; its advantages will be shown later. In this study, the load will be taken as 1½ kips, placed at the panel point just to the right of the center; this is Panel Point (9).

From Plate VI and Figs. 49 and 50 are found the moments of inertia, the centers of gravity of sections, the distances to the extreme fibers, the areas of cross-section and the lengths of the members. None of the calculations is reproduced, only the results are given. It should be remembered that E, the modulus of elasticity, may be regarded as unity throughout the investigation.

The work of calculating the secondary stresses in the trusses when the bridge is under a large number of concentrated loads, will not be appreciably greater than when a single load is considered. The secondary stresses depend on the set of elongations of all the members considering the bridge as statically determinate. The actual work of determining the secondary stresses from the set of elongations under many loads has, therefore, exactly the same weight as for a single load; there is, of course, a little more preliminary work necessary in calculating the elongations.

As was stated previously, the exact method of Manderla (see page 976) is one that involves a great amount of work, and gives results that differ little from those obtained by the modified method. This modified method is generally known as Manderla's method. No solution is made by the exact method; the method, designated "Manderla" in this comparison, is the modified form, due to Winkler. The whole investigation was facilitated by the use of a calculating machine.

2.—Application of Methods

Manderla's Method.—In order to avoid repetition, only one computation was prepared to illustrate the two methods of setting about a solution. The first method would contain the solution of a set of simultaneous linear equations by means of the method of substitution, while the second would solve a set of simultaneous linear equations by means of Gauss' theory of normal equations. Otherwise, the first method would be similar to the second. Only the results of the first method are given in Table 24.

Manderla-Gauss Method.—The formation of the equations for the truss and loading shown in Fig. 51 and Plate VI, is given in Table 1. The re-

TABLE 1.—Formation of Equations.

1.— $-15.930\,\tau_1 + 5.464\,\tau_2 + 2.501\,\tau_3 + 80.58 - 52.62 = 0$

52.62
27.96

2.— $-30.806\,\tau_2 + 7.951\,\tau_4 + 1.685\,\tau_5 + 0.303\,\tau_3 + 5.464\,\tau_1 - 216.08 + 270.89 = 0$

45.61
11.78
273.47
270.89
2.58

3.— $-10.610\,\tau_3 + 2.501\,\tau_1 + 0.303\,\tau_2 + 2.501\,\tau_5 - 55.98 + 40.29 = 0$

9.81
65.79
40.29
25.50

4.— $-32.184\,\tau_4 + 7.951\,\tau_6 + 0.190\,\tau_5 + 7.951\,\tau_2 + 532.04 - 11.34 = 0$

308.02
840.06
11.34
828.72

5.— $-19.528\,\tau_5 + 2.501\,\tau_3 + 1.685\,\tau_2 + 0.190\,\tau_4 + 1.800\,\tau_6 + 3.588\,\tau_7 - 569.62 + 25.43 = 0$

16.21
45.61
4.87
636.31
25.43
610.88

6.— $-39.610\,\tau_6 + 7.951\,\tau_8 + 1.800\,\tau_9 + 0.303\,\tau_7 + 1.800\,\tau_5 + 7.951\,\tau_4 + 565.42 - 44.30 = 0$

1 510.93	18.18
2 076.35	105.62
168.10	168.10
1 908.25	

7.— $-14.958\,\tau_7 + 3.588\,\tau_5 + 0.303\,\tau_6 + 3.588\,\tau_9 - 314.36 = 0$

122.24
6.44
443.04

8.— $-19.528\,\tau_9 + 3.588\,\tau_7 + 1.800\,\tau_6 + 0.190\,\tau_8 + 1.685\,\tau_{10} + 2.501\,\tau_{11} - 1\,523.90 + 22.68 = 0$

139.00	58.76
44.30	81.44
1 707.20	
81.44	
1 625.76	

9.— $-32.184\,\tau_8 + 7.951\,\tau_{10} + 0.190\,\tau_9 + 7.951\,\tau_6 + 3\,067.22 - 153.30 = 0$

166.72	13.42
2 900.50	166.72

TABLE 1.—(*Continued*).

$$10.\quad -10.610\,\tau_{11} + 2.501\,\tau_9 + 0.303\,\tau_{10} + 2.501\,\tau_{12} - 25.46 + 13.79 = 0$$

$$\begin{array}{r} 475.27 \\ \hline 500.73 \\ 13.79 \\ \hline 486.94 \\ \hline \end{array}$$

$$11.\quad -30.806\,\tau_{10} + 5.464\,\tau_{12} + 0.303\,\tau_{11} + 1.685\,\tau_9 + 7.951\,\tau_8 - 161.50 + 176.05 = 0$$

$$\begin{array}{rr} 228.96 & 19.63 \\ \hline 390.46 & 195.68 \\ 195.68 & \\ \hline 194.78 & \\ \hline \end{array}$$

$$12.\quad -15.930\,\tau_{12} + 2.501\,\tau_{11} + 5.464\,\tau_{10} + 352.10 - 32.36 = 0$$

$$\begin{array}{r} 32.36 \\ \hline 319.74 \\ \hline \end{array}$$

mainder of the solution of the secondary stresses according to Gauss' theory of normal equations is given in Tables 2 to 7.

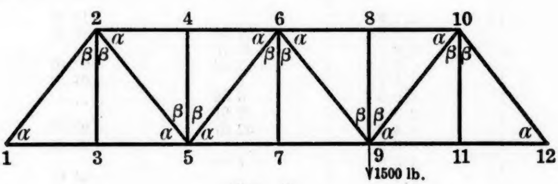


FIG. 51.

TABLE 2.—MANDERLA-GAUSS METHOD. ELEMENT OF MEMBERS.

Member.	Sectional area, in square inches.	Length, l, in inches.	Primary stress, in pounds.	Unit stress, in pounds.	I, in inches^4.	$K=\frac{I}{l}$.	Make-up of member.
1- 3 11-12	26.48	300	+ 403.2 + 806.5	+15.23 +30.46	750.2	2.501	Two 15-in. angles, 45 lb.
3- 5 9-11	26.48	300	+ 403.2 + 806.5	+15.23 +30.46	750.2	2.501	Two 15-in. angles, 45 lb.
5- 7 7- 9	50.36	300	+1 209.7 +1 209.7	+24.02 +24.02	1 076.4	3.588	Two 15-in. angles, 55 lb. Two plates. 12 in. × ¾ in.
1- 2 10-12	49.45	477.9	− 642.3 −1 284.7	−12.99 −25.98	2 611.5	5.464	One cover plate, 21 in. × ⅜ in.; Two webs 18 in. × ⅝ in. Two angles 3½ × 3½ × 7/16 in.; Two angles 5 × 3½ × ⅞ in.
2- 4 8-10	43.33	300	− 806.5 −1 612.9	−18.61 −37.22	2 385.4	7.951	One cover plate, 21 in. × ⅜ in.; Two webs 18 in. × ½ in. Two angles 3½ × 3½ × ⅜ in.; Two angles 5 × 3½ × 13/16 in.
4- 6 6- 8	43.33	300	− 806.5 −1 612.9	−18.61 −37.22	2 385.4	7.951	One cover plate 21 in. × ⅜ in.; Two webs 18 in. × ½ in. Two angles 3½ × 3½ × ⅜ in.; Two angles 5 × 3½ × 13/16 in.
2- 3 6- 7 10-11	24.35	372	....	....	112.86	0.303	Four angles 5 × 3½ × 9/16 in. One web 11½ × 9/16 in.
4- 5 8- 9	12.20	372	....	....	70.62	0.190	Four angles 5 in. × 3½ × ⅜ in.
2- 5 9-10	29.42	477.9	+ 642.3 +1 284.7	+21.83 +43.67	805.4	1.685	Two 15-in. angles, 50 lb.
5- 6 6- 9	32.36	477.9	− 642.3 + 642.3	−19.85 +19.85	860.4	1.800	Two 15-in. angles, 55 lb.

Table 2 is obtained from the dimensions of the truss and is self-explanatory. Use is then made of Equations (2), namely,

$$E \,.\, \varDelta\, \alpha = (s_1 - s_2)\; cot\; \gamma + (s_1 - s_3)\; cot\; \beta$$

$$E \,.\, \varDelta\, \beta = (s_2 - s_3)\; cot\; \alpha + (s_2 - s_1)\; cot\; \gamma$$

$$E \,.\, \varDelta\, \gamma = (s_3 - s_1)\; cot\; \beta + (s_3 - s_2)\; cot\; \alpha$$

to obtain Table 3.

The method of constructing Tables 3 and 4 and hence of forming the expression for the equality of the moments acting at each joint to zero are given in standard works on structural design.*

A little practice will enable the engineer to perform the latter operation mechanically.

TABLE 3.—Manderla-Gauss Method. Angles and Functions.

Triangle.	Angle.	cot. α = 0.8065	cot. β = 1.24	$\dfrac{E\ \delta\ \text{angle}}{10^6}$
2.1.3	2.1.3	....	+16.11	+16.11
	1.3.2	−22.76	−16.11	−38.87
	3.2.1	+22.76	....	+22.76
2.3.5	2.3.5	+ 5.32	+27.07	+32.39
	3.5.2	....	−27.07	−27.07
	5.2.3	− 5.32	......	− 5.32
2.5.4	2.5.4	−32.61	......	−32.61
	5.4.2	+32.61	+27.07	+59.68
	4.2.5	....	−27.07	−27.07
4.5.6	4.5.6	+ 1.00	......	+ 1.00
	5.6.4	....	+24.61	+24.61
	6.4.5	− 1.00	−24.61	−25.61
5.7.6	5.7.6	−35.38	−24.61	−59.99
	7.6.5	+35.38	......	+35.38
	6.5.7	....	+24.61	+24.61
6.7.9	6.7.9	− 3.36	+24.61	+21.25
	7.9.6	....	−24.61	−24.61
	9.6.7	+ 3.36	......	+ 3.36
6.9.8	6.9.8	−46.03	......	−46.03
	9.8.6	+46.03	+24.61	+70.64
	8.6.9	....	−24.61	−24.61
8.9.10	8.9.10	−65.24	......	−65.24
	9.10.8	....	−54.15	−54.15
	10.8.9	+65.24	+54.15	+119.39
9.11.10	9.11.10	+10.65	+54.15	+64.80
	11.10.9	−10.65	......	−10.65
	10.9.11	....	−54.15	−54.15
10.11.12	10.11.12	−45.52	−32.22	−77.74
	11.12.10	....	+32.22	+32.22
	12.10.11	+45.52	......	+45.52

Table 5 is then constructed. It represents the solution of the twelve equations already formed; Gauss' method of normal equations is used. Textbooks on least squares explain the theory; the remarks in the second column ought, however, to be sufficient to indicate the reasoning. The check term in the last column is the sum of all the coefficients of the τ's in an equation, together with its absolute term. Thus, $+\ 15.930 + 5.464 + 2.501 + 2.796 = +\ 26.691$ for Equation (1). All the absolute terms are divided by 10, in order to obtain the three decimal places of the τ-coefficients. The resulting τ's are, therefore, to be multiplied by 10, before being used in Table 5.

* See Johnson, Bryan, and Turneaure, "Theory of Framed Structures."

TABLE 4.—MANDERLA-GAUSS METHOD. CHANGES OF ANGLE AND SUMMATIONS.

Joint.	Member.	$K = \frac{I}{l}$.	Angle.	δ Angle.	$\Sigma\,\delta$ Angle.	$K\,\Sigma\,\delta$ Angle.
1	1-2	5.464	....	....	....	....
	1-3	2.501	2.1.3	+16.11	+16.11	+40.29
	.....	7.965	....	....	....	+40.29
2	2-4	7.951	....	....	....	....
	2-5	1.685	4.2.5	—27.07	—27.07	—45.61
	2-3	0.303	5.2.3	— 5.32	—32.39	— 9.81
	2-1	5.464	3.2.1	+22.76	— 9.63	—52.62
	.....	15.403	....	....	....	—108.04
3	3-1	2.501	....	....	....	....
	3-2	0.303	1.3.2	—38.87	—38.87	—11.78
	3-5	2.501	2.3.5	+32.39	— 6.48	—16.21
	.....	5.305	....	....	....	—27.99
4	4-6	7.951	....	....	....	....
	4-5	0.190	6.4.5	—25.61	—25.61	— 4.87
	4-2	7.951	5.4.2	+59.68	+34.07	+270.89
	.....	16.092	....	....	....	+266.02
5	5-3	2.501	....	....	....	....
	5-2	1.685	3.5.2	—27.07	—27.07	—45.61
	5-4	0.190	2.5.4	—32.61	—59.68	—11.34
	5-6	1·800	4.5.6	+ 1.00	—58.68	—105.62
	5-7	3.588	6.5.7	+24.61	—34.07	—122.24
	.....	9.764	....	....	....	—284.81
6	6-8	7.951	....	....	....	....
	6-9	1.800	8.6.9	—24.61	—24.61	—44.30
	6-7	0.303	9.6.7	+ 3.36	—21.25	— 6.44
	6-5	1.800	7.6.5	+35.38	+14.13	+25.43
	6-4	7.951	5.6.4	+24.61	+38.74	+308.02
	.....	19.805	....	....	....	+282.71
7	7-5	3.588	....	....	....	....
	7-6	0.303	5.7.6	—59.99	—59.99	—18.18
	7-9	3.588	6.7·9	+21.25	—38.74	—139.00
	.....	7.479	....	....	....	—157.18
8	8-10	7.951	....	....	....	....
	8-9	0.190	10.8.9	+119.39	+119.39	+22.68
	8-6	7.951	9.8.6	+70.64	+190.03	+1510.93
	.....	16.092	....	....	....	+1533.61
9	9-7	3.588	....	....	....	....
	9-6	1.800	7.9.6	—24.61	—24.61	—44.30
	9-8	0.190	6.9.8	—46.03	—70.64	—13.42
	9-10	1.685	8.9.10	—65.24	—135.88	—228.96
	9-11	2.501	10.9.11	—54.15	—190.03	—475.27
	.....	9.764	....	....	....	—761.95
10	10-12	5.464	....	....	....	....
	19-11	0.303	12.10.11	+45.52	+45.52	+13.79
	10-9	1.685	11.10.9	—10.65	+34.87	+58.76
	10-8	7.951	9.10.8	—54.15	—19.28	—153.30
	.....	15.403	....	....	....	—80.75
11	11-9	2.501	....	....	....	....
	11-10	0.303	9.11.10	+64.80	+64.80	+19.63
	11-12	2.501	10.11.12	—77.74	—12.94	—32.36
	.....	5.305	....	....	....	—12.73
12	12-11	2.501	....	....	....	....
	12-10	5.464	11.12.10	+32.22	+32.22	+176.05
	.....	7.965	....	....	....	+176.05

When the operations on Equations (1) and (2), Table 5, have been performed, Equation I is obtained by adding the two equations above. The check terms also are added. The final check for Equation I is that the sum of the coefficients of its τ's, together with its absolute term, must give the check term already found by adding -9.155 and $+45.951$, namely, $+36.796$. This check is applied for Equations II, III, IV, etc., until finally Equation XI is reached, which gives the value for τ_{12}. Going back to Equations X, IX, VIII, etc., τ_{10}, τ_{11}, τ_8, etc., are solved in turn.

These values are now inserted in Table 6, and by use of Equation (27), the τ's of all the members are obtained. For example, $\tau_{12} = -2.16$. In order to get the τ of the next member, namely, 1-3, it is necessary to add the δ-angle of

TABLE 5.—MANDERLA-GAUSS METHOD. VALUES FOR EQUATIONS.

Number of equation.	Remarks.	τ_1.	τ_2.	τ_3.	τ_4.	τ_5.	τ_6.	τ_7.	τ_9.	τ_8.	τ_{11}.	τ_{10}.	τ_{12}.	Absolute term.	Check.
1	....	+15.930	+5.464	+2.501	......	......	......	......	......	......	......	......	......	+2.796	+26.691
1′	$1 \times -\frac{5.464}{15.930}$, *i. e.*, 0.34300	[−5.464]	−1.874	−0.858	......	......	......	......	......	......	......	......	......	−0.959	−9.155
2	....	[+5.464]	+30.806	+0.303	+7.951	+1.685	......	......	......	......	......	......	......	−0.258	+45.951
I	1′ + 2	......	+28.932	−0.555	+7.951	+1.685	......	......	......	......	......	......	......	−1.217	+36.796
1″	$1 \times -\frac{2.501}{15.930}$, *i. e.*, 0.15700	[−2.501]	[−0.858]	−0.393	......	......	......	......	......	......	......	......	......	−0.439	−4.190
I′	$\mathrm{I} \times +\frac{0.555}{28.932}$, *i. e.*, 0.01918	......	[+0.555]	−0.011	+0.153	+0.032	......	......	......	......	......	......	......	−0.023	+0.706
3	....	[+2.501]	[+0.303]	+10.610	......	+2.501	......	......	......	......	......	......	......	−2.550	+13.365
II	1″ + I′ + 3	......	......	+10.206	+0.153	+2.533	......	......	......	......	......	......	......	−3.012	+9.881
I″	$\mathrm{I} \times -\frac{7.951}{28.932}$, *i. e.*, 0.27482	......	[−7.951]	[+0.153]	−2.185	−0.463	......	......	......	......	......	......	......	+0.334	−10.112
II′	$\mathrm{II} \times -\frac{0.153}{10.206}$, *i. e.*, 0.01499	......	......	[−0.153]	−0.002	−0.038	......	......	......	......	......	......	......	+0.045	−0.148
4	....	......	[+7.951]	......	+32.184	+0.190	+7.951	......	......	......	......	......	......	+82.872	+131.148
III	I″ + II′ + 4	......	......	......	+29.997	−0.311	+7.951	......	......	......	......	......	......	+83.251	+120.888
I‴	$\mathrm{I} \times -\frac{1.685}{28.932}$, *i. e.*, 0.05824	......	[−1.685]	[+0.032]	[−0.463]	−0.098	......	......	......	......	......	......	......	+0.071	−2.143
II″	$\mathrm{II} \times -\frac{2.533}{10.206}$, *i. e.*, 0.24819	......	......	[−2.533]	[−0.038]	−0.629	......	......	......	......	......	......	......	+0.748	−2.452
III′	$\mathrm{III} \times +\frac{0.311}{29.997}$, *i. e.*, 0.01037	......	......	......	[+0.311]	−0.003	+0.082	......	......	......	......	......	......	+0.863	+1.253
5	....	......	[+1.685]	[+2.501]	[+0.190]	+19.528	+1.800	+3.588	......	......	......	......	......	−61.088	−31.796
IV	I‴ + II″ + III′ + 5	......	......	......	......	+18.798	+1.882	+3.588	......	......	......	......	......	−59.406	−35.188

TABLE 5.—(*Continued*).

Number of equation.	Remarks.	τ_1.	τ_2.	τ_3.	τ_4.	τ_5.	τ_6.	τ_7.	τ_9.	τ_8.	τ_{11}.	τ_{10}.	τ_{12}.	Absolute term.	Check.
III″	III $\times -\frac{7.951}{29.997}$, *i. e.*, 0.26506	......	......	......	[−7.951]	[+0.082]	−2.107	......	......	......	......	......	......	−22.067	−32.043
IV′	IV $\times -\frac{1.882}{18.798}$, *i. e.*, 0.10012	......	......	......	......	[−1.882]	−0.188	−0.359	......	......	......	......	......	+5.947	+3.518
6	......	......	......	......	[+7.951]	[+1.800]	+39.610	+0.303	+1.800	+7.951	......	......	......	+190.825	+250.240
V	III″ + IV′ + 6	......	......	......	......	......	+37.315	−0.056	+1.800	+7.951	......	......	......	+174.705	+221.715
IV″	IV $\times -\frac{3.588}{18.798}$, *i. e.*, 0.19087	......	......	......	......	[−3.588]	[−0.359]	−0.685	......	......	......	......	......	+11.339	+6.707
V′	V $\times +\frac{0.056}{37.315}$, *i. e.*, 0.00150	......	......	......	......	......	[+0.056]	−0.000	+0.003	+0.012	......	......	......	+0.262	+0.333
7	......	......	......	......	......	[+3.588]	[+0.303]	+14.958	+3.588	......	......	......	......	−44.304	−21.867
VI	IV″ + V′ + 7	......	......	......	......	......	......	+14.273	+3.591	+0.012	......	......	......	−32.703	−14.827
V″	V $\times -\frac{1.800}{37.315}$, *i. e.*, 0.04824	......	......	......	......	......	[−1.800]	[+0.003]	−0.087	−0.384	......	......	......	−8.427	−10.695
VI′	VI $\times -\frac{3.591}{14.273}$, *i. e.*, 0.25159	......	......	......	......	......	......	[−3.591]	−0.903	−0.003	......	......	......	+8.228	+3.730
8	......	......	......	......	......	......	[+1.800]	[+3.588]	+19.528	+0.190	+2.501	+1.685	......	−162.576	−133.284
VII	V″ + VI′ + 8	......	......	......	......	......	......	......	+18.538	−0.197	+2.501	+1.685	......	−162.775	−140.249
V‴	V $\times -\frac{7.951}{37.315}$, *i. e.*, 0.21308	......	......	......	......	......	[−7.951]	[+0.012]	[−0.384]	−1.694	......	......	......	−37.226	−47.243
VI″	VI $\times -\frac{0.012}{14.273}$, *i. e.*, 0.00084	......	......	......	......	......	......	[−0.012]	[−0.003]	−0.000	......	......	......	+0.027	+0.012
VII′	VII $\times +\frac{0.197}{18.538}$, *i. e.*, 0.01063	......	......	......	......	......	......	......	[+0.197]	−0.012	+0.027	+0.018	......	−1.730	−1.490
9	......	......	......	......	......	......	[+7.951]	......	[+0.190]	+32.184	......	+7.951	......	+290.050	+338.326
VIII	V‴ + VI″ + VII′ + 9	......	......	......	......	......	......	......	......	+30.488	+0.027	+7.969	......	+251.121	+289.605

TABLE 5.—(*Continued*).

Number of equation.	Remarks.	τ_1.	τ_2.	τ_3.	τ_4.	τ_5.	τ_6.	τ_7.	τ_9.	τ_8.	τ_{11}.	τ_{10}.	τ_{12}.	Absolute term.	Check.
VII″	VII $\times -\frac{2.501}{18.538}$, *i. e.*, 0.13491	......	......	......	......	......	......	......	[−2.501]	[+0.027]	−0.337	−0.227	......	+21.960	+18.921
VIII′	VIII $\times -\frac{0.027}{30.488}$, *i. e.*, 0.00089	......	......	......	......	......	......	......	......	[−0.027]	−0.000	−0.007	......	−0.222	−0.256
10	....	......	......	......	......	......	......	......	[+2.501]	......	+10.610	+0.303	+2.501	−48.694	−32.779
IX	VII″ + VIII′ + 10	......	......	......	......	......	......	......	......	......	+10.273	+0.069	+2.501	−26.956	−14.114
VII‴	VII $\times -\frac{1.685}{18.538}$, *i. e.*, 0.09089	......	......	......	......	......	......	......	[−1.685]	[+0.018]	[−0.227]	−0.153	......	+14.795	+12.748
VIII″	VIII $\times -\frac{7.969}{30.488}$, *i. e.*, 0.26138	......	......	......	......	......	......	......	......	[−7.969]	[−0.007]	−2.083	......	−65.638	−75.697
IX′	IX $\times -\frac{0.069}{10.273}$, *i. e.*, 0.00672	......	......	......	......	......	......	......	......	......	[−0.069]	−0.000	−0.017	+0.181	+0.095
11	....	......	......	......	......	......	......	......	[+1.685]	[+7.951]	[+0.303]	+30.806	+5.464	−19.478	+26.731
X	VII‴ + VIII″ + IX′ + 11	......	......	......	......	......	......	......	......	......	......	+28.570	+5.447	−70.140	−36.123
IX″	IX $\times -\frac{2.501}{10.273}$, *i. e.*, 0.24345	......	......	......	......	......	......	......	......	......	[−2.501]	[−0.017]	−0.609	+6.563	+3.436
X′	X $\times -\frac{5.447}{28.570}$, *i. e.*, 0.19065	......	......	......	......	......	......	......	......	......	......	[−5.447]	−1.038	+13.372	+6.887
12	....	......	......	......	......	......	......	......	......	......	[+2.501]	[+5.464]	+15.930	+31.974	+55.869
XI	IX″ + X′ + 12	......	......	......	......	......	......	......	......	......	......	......	+14.283	+51.909	+66.192
		−0.21599	+0.35784	−0.52401	−1.90951	+3.41579	−3.13281	+0.30432	+7.92766	−9.06261	+3.48761	+3.14792	−3.63432	....	....

TABLE 6.—Manderla-Gauss Method. Solution for Terms, T and M.

1	τ_{12}	$= -2.16$	$M_{12} = 5.464\,(-4.32 - 6.05) \times 2 = -113.32$ in.-lb.
		$+16.11$	
	τ_{13}	$= +13.95$	$M_{13} = 2.501\,(+27.90 - 5.24) \times 2 = +113.35$ in.-lb.
2	τ_{24}	$= +3.58$	$M_{24} = 7.951\,(+7.16 + 14.97) \times 2 = +351.91$ in.-lb.
		-27.07	
	τ_{25}	$= -23.49$	$M_{25} = 1.685\,(-46.98 + 7.09) \times 2 = -134.43$ in.-lb.
		-5.32	
	τ_{23}	$= -28.81$	$M_{23} = 0.303\,(-57.62 - 44.11) \times 2 = -61.65$ in.-lb.
		$+22.76$	
	τ_{21}	$= -6.05$	$M_{21} = 5.464\,(-12.10 - 2.16) \times 2 = -155.83$ in.-lb.
3	τ_{31}	$= -5.24$	$M_{31} = 2.501\,(-10.48 + 13.95) \times 2 = +17.36$ in.-lb.
		-38.87	
	τ_{32}	$= -44.11$	$M_{32} = 0.303\,(-88.22 - 28.81) \times 2 = -70.92$ in.-lb.
		$+32.39$	
	τ_{35}	$= -11.72$	$M_{35} = 2.501\,(-23.44 + 34.16) \times 2 = +53.62$ in.-lb.
4	τ_{46}	$= -19.10$	$M_{46} = 7.951\,(-38.20 + 7.41) \times 2 = -489.62$ in.-lb.
		-25.61	
	τ_{45}	$= -44.71$	$M_{45} = 0.190\,(-89.42 - 25.52) \times 2 = -43.68$ in.-lb.
		$+59.68$	
	τ_{42}	$= +14.97$	$M_{42} = 7.951\,(+29.94 + 3.58) \times 2 = +533.04$ in.-lb.
5	τ_{53}	$= +34.16$	$M_{53} = 2.501\,(+68.32 - 11.72) \times 2 = +283.11$ in.-lb.
		-27.07	
	τ_{52}	$= +7.09$	$M_{52} = 1.685\,(+14.18 - 23.49) \times 2 = -31.37$ in.-lb.
		-32.61	
	τ_{54}	$= -25.52$	$M_{54} = 0.190\,(-51.04 - 44.71) \times 2 = -36.38$ in.-lb.
		$+1.00$	
	τ_{56}	$= -24.52$	$M_{56} = 1.800\,(-49.04 - 17.20) \times 2 = -238.46$ in.-lb.
		$+24.61$	
	τ_{57}	$= +0.09$	$M_{57} = 3.588\,(+0.18 + 3.04) \times 2 = +23.11$ in.-lb.
6	τ_{68}	$= -31.33$	$M_{68} = 7.951\,(-62.66 + 99.40) \times 2 = +584.24$ in.-lb.
		-24.61	
	τ_{69}	$= -55.94$	$M_{69} = 1.800\,(-111.88 + 54.67) \times 2 = -205.96$ in.-lb.
		$+3.36$	
	τ_{67}	$= -52.58$	$M_{67} = 0.303\,(-105.16 - 56.95) \times 2 = -98.24$ in.-lb.
		$+35.38$	
	τ_{65}	$= -17.20$	$M_{65} = 1.800\,(-34.40 - 24.52) \times 2 = -212.11$ in.-lb.
		$+24.61$	
	τ_{64}	$= +7.41$	$M_{64} = 7.951\,(+14.82 - 19.10) \times 2 = -68.06$ in.-lb.
7	τ_{75}	$= +3.04$	$M_{75} = 3.588\,(+6.08 + 0.09) \times 2 = +44.28$ in.-lb.
		-59.99	
	τ_{76}	$= -56.95$	$M_{76} = 0.303\,(-113.90 - 52.58) \times 2 = -100.89$ in.-lb.
		$+21.25$	
	τ_{79}	$= -35.70$	$M_{79} = 3.588\,(-71.40 + 79.28) \times 2 = +56.55$ in.-lb.
8	$\tau_{8.10}$	$= -90.63$	$M_{8.10} = 7.951\,(-181.26 + 12.20) \times 2 = -2\,688.39$ in.-lb.
		$+119.39$	
	τ_{89}	$= +28.76$	$M_{89} = 0.190\,(+57.52 + 8.64) \times 2 = +25.14$ in.-lb.
		$+70.64$	
	τ_{86}	$= +99.40$	$M_{8.6} = 7.951\,(+198.80 - 31.33) \times 2 = +2\,663.11$ in.-lb.

TABLE 6.—(*Continued*).

9	$\tau_{97} = +79.28$ -24.61	$M_{97} = 3.588\ (+158.56 - 35.70) \times 2 = +881.64$ in.-lb.
	$\tau_{9.6} = +54.67$ -46.03	$M_{9.6} = 1.800\ (+109.34 - 55.94) \times 2 = +192.24$ in.-lb.
	$\tau_{9.8} = +8.64$ -65.24	$M_{9.8} = 0.190\ (+17.28 + 28.76) \times 2 = +17.50$ in.-lb.
	$\tau_{9.10} = -56.60$ -54.15	$M_{9.10} = 1.685\ (-113.20 + 66.35) \times 2 = -157.88$ in.-lb.
	$\tau_{9.11} = -110.75$	$M_{9.11} = 2.501\ (-221.50 + 34.88) \times 2 = -933.47$ in.-lb.
10	$\tau_{10.12} = +31.48$ $+45.52$	$M_{10.12} = 5.464\ (+62.96 - 4.12) \times 2 = +643.00$ in.-lb.
	$\tau_{10.11} = +77.00$ -10.65	$M_{10.11} = 0.303\ (+154.00 + 99.68) \times 2 = +153.73$ in.-lb.
	$\tau_{10.9} = +66.35$ -54.15	$M_{10.9} = 1.685\ (+132.70 - 56.60) \times 2 = +256.46$ in.-lb.
	$\tau_{10.8} = +12.20$	$M_{10.8} = 7.951\ (+24.40 - 90.63) \times 2 = -1\,053.19$ in.-lb.
11	$\tau_{11.9} = +34.88$ $+64.80$	$M_{11.9} = 2.501\ (+69.76 - 110.75) \times 2 = -205.03$ in.-lb.
	$\tau_{11.10} = +99.68$ -77.74	$M_{11.10} = 0.303\ (+199.36 + 77.00) \times 2 = +167.47$ in.-lb.
	$\tau_{11.12} = +21.94$	$M_{11.12} = 2.501\ (+43.88 - 36.34) \times 2 = +37.72$ in.-lb.
12	$\tau_{12.11} = -36.34$ $+32.22$	$M_{12.11} = 2.501\ (-72.68 + 21.94) \times 2 = -253.80$ in.-lb.
	$\tau_{12.10} = -4.12$	$M_{12.10} = 5.464\ (-8.24 + 31.48) \times 2 = +253.97$ in.-lb.

the angle between the members; this is found from Table 4 to be + 16.11. The bending moments are then obtained by making use of Equations (39). A check is applied at this juncture; the sum of the bending moments around a joint must be zero.

Table 7 then follows directly. It is to be noted that when a member like 1-3 is subjected to the bending moment, + 113.35 in-lb., positive in a counter-clockwise direction, the top set of fibers of Member 1-3, at End (1) is in tension and the bottom is in compression. In other words, the quantities, — 0.41, + 0.41, ± 1.13, in the last column of Table 7, Joint (1), are the secondary stresses of the top set of Member 1-2, the bottom set of Member 1-2, the top set of Member 1-3, and the bottom set of Member 1-3, in the order named, which is also the order due to a clockwise movement around Joint (1). The quantities, — 0.41 and + 0.41, are separated because the distance of the top set of fibers from the neutral plane, through the center of gravity, differs from that of the bottom set.

Müller-Breslau's Method.—The work of solving the problem is given in Tables 8 to 10, inclusive, and Fig. 52. For this solution the K's and the changes of angle were just taken from Table 4. The ρ's, which are the $\frac{M\,l}{I}$ quantities, are written at both ends of all the members. Thus, ρ_1 to ρ_{42}, inclusive, are obtained. Six times the changes of angle are the quantities found in the corners.

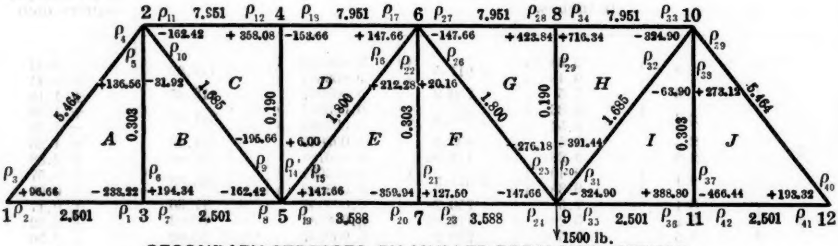
TABLE 7.—MANDERLA-GAUSS METHOD. SECONDARY STRESSES AND MEMBERS.

Joint.	Member.	Extreme fiber distance, C, in inches.	I, in inches[4].	$\frac{C}{I}$	M, in pounds.	$f = \frac{C}{I} \times M$, in pounds per square inch.
1	1-2	9.38 9.49	2 611.5	0.003592 0.003634	− 113.32	− 0.41 + 0.41
	1-3	7.5	750.2	0.009997	+ 113.35	± 1.13
2	2-4	9.33 9.54	2 385.4	0.003911 0.003999	+ 351.91	+ 1.38 − 1.41
	2-5	7.5	805.4	0.009312	− 134.43	∓ 1.25
	2-3	5-28	112.86	0.046784	− 61.65	∓ 2.88
	2-1	9.49 9.38	2 611.5	0.003634 0.003592	− 155.83	− 0.57 + 0.56
3	3-1	7.5	750.2	0.009997	+ 17.36	± 0.17
	3-2	5.28	112.86	0.046784	− 70.92	∓ 3.32
	3-5	7.5	750.2	C.009997	+ 53.62	± 0.56
4	4-6	9.33 9.54	2 385.4	0.003911 0.003999	− 489.62	− 1.92 + 1.96
	4-5	5.19	70.62	0.073492	− 43.68	∓ 3.21
	4-2	9.54 9.33	2 385.4	0.003999 0.003911	+ 533.04	+ 2.13 − 2.08
5	5-3	7.5	750.2	0.009997	+ 283.11	± 2.83
	5-2	7.5	805.4	0.009312	− 31.37	∓ 0.29
	5-4	5.19	70.62	0.073492	− 36.38	∓ 2.70
	5-6	7.5	860.4	0.008717	− 238.46	∓ 2.08
	5-7	7.5	1 076.4	0-006968	+ 23.11	± 0.16
6	6-8	9.33 9.54	2 385.4	0.003911 0.003999	+ 584.24	+ 2.28 − 2.34
	6-9	7.5	860.4	0.008717	− 205.96	∓ 1.80
	6-7	5.28	112.86	0.046784	− 98.24	∓ 4.60
	6-5	7.5	860.4	0.008717	− 212.11	∓ 1.85
	6-4	9.54 9.33	2 385.4	0.003999 0.003911	− 68.06	− 0.27 + 0.27
7	7-5	7.5	1 076.4	0.006968	+ 44.28	± 0.31
	7-6	5.28	112.86	0.046784	− 100.89	∓ 4.72
	7-9	7.5	1 076.4	0.006968	+ 56.55	± 0.39
8	8-10	9.33 9.54	2 385.4	0.003911 0.003999	−2 688.39	−10.52 +10.75
	8-9	5.19	70.62	0.073492	+ 25.14	± 1.85
	8-6	9.54 9.33	2 385.4	0.003999 0.003911	+2 663.11	+10.65 −10.42
9	9-7	7.5	1 076.4	0.006968	+ 881.64	± 6.14
	9-6	7.5	860.4	0.008717	+ 192.24	± 1.68
	9-8	5.19	70.62	0.073492	+ 17.50	± 1.29
	9-10	7.5	805.4	0.009312	− 157.88	∓ 1.47
	9-11	7.5	750.2	0.009997	− 933.47	∓ 9.33
10	10-12	9.38 9.49	2 611.5	0.003592 0.003634	+ 643.00	+ 2.31 − 2.34
	10-11	5.28	112.86	0.046784	+ 153.73	± 7.19
	10-9	7.5	805.4	0.009312	+ 256.46	± 2.39
	10-8	9.54 9.33	2 385.4	0.003999 0.003911	−1 053.19	− 4.21 + 4.12
11	11-9	7.5	750.2	0.009997	− 205.03	∓ 2.05
	11-10	5.28	112.86	0.046784	+ 167.47	± 7.83
	11-12	7.5	750.2	0.009997	+ 37.72	± 0.38
12	12-11	7.5	750.2	0.009997	− 253.80	∓ 2.54
	12-10	9.49 9.38	2 611.5	0.003634 0.003592	+ 253.97	+ 0.92 − 0.91

It is assumed that the two quantities, ρ_1 and ρ_2, on Ends (3) and (1) of the member, 1-3, respectively, are known. $\Sigma M = 0$ for Joint (1), that is, $K_{12} \cdot \rho_3 + K_{13} \cdot \rho_2 = 0$, expressing ρ_3 in terms of ρ_2. Three of the six quantities, ρ_1, ρ_2, ρ_3, ρ_4, ρ_5, and ρ_6, therefore, are known. Applying Equations (47), (48), and (49), ρ_4, ρ_5, and ρ_6 are given in terms of ρ_1 and ρ_2 as shown.

$\Sigma M = 0$ for Joint (3) gives an expression for ρ_7 in terms of ρ_1 and ρ_2. This process is continued until the $\Sigma M = 0$ equations for Joints (10) and (12) give the solution of ρ_1 and ρ_2. On substituting these values, ρ_3, ρ_4, ρ_5, * * * ρ_{42}, are obtained as shown in Table 9. Table 10 is self-explanatory. Note that a clockwise moment was assumed to be positive at the start (see

page 987). This means that the top fiber of the member, 1-3, End (1), is + 1.14, the result of the ρ_2 being — 45.60. This agrees with the signs of the fiber stresses found in Table 7.



SECONDARY STRESSES BY MULLER-BRESLAU'S METHOD

REMARKS.

The numbers written on the members are the $K = I/l$ - Values; e. g. 5.464 on 1-2 = K_{12}

The numbers written in the corners are the Changes of Angle, multiplied by 6E: e. g. +96.66 in corner of Triangle $A = 6E.\ \delta A1.$

FIG. 52.

Ritter's Method.—In Fig. 53 and Tables 11 and 12 are given the results of applying Ritter's graphical method to a solution of the truss. The unit stresses are again assumed to be known, and with these the changes of angle (see page 977) are determined graphically as indicated in Fig. 53 and Table 11.

The quantities, $\frac{I}{l}$, are the forces of the force and equilibrium polygons. As already described (page 992), these forces are laid out, spaced by horizontal distances equal to the $E\,.\,\Delta\,\theta$'s of Table 11. This gives 3, 5, 1, 4, in order from left to right for Joint (2). The scale used is of importance. Thus (Fig. 53 (2)), 5 is at a distance corresponding to 2.72 to the left of 4, representing the change of angle, $C\,2 = -27.2$, in Table 11. When all these forces have been laid out, draw the force polygons at the sides, with the forces in the order met in going from left to right along the horizontal. Thus, the force polygon of Joint (2) has the forces, 3, 5, 1, 4, in the order named, Force 3 meaning $\frac{I_{23}}{l_{23}}$, etc. The scale of the force polygons is arbitrary.

Draw the equilibrium polygons for the undisplaced forces, obtaining the positions of their resultants. By means of Equations (53), are found the distances to which all these forces have to be displaced, and then the resultant of these forces by means of a second set of equilibrium polygons. The resultants just found will be displaced from the previous resultants. Again, use Equations (53) to obtain a fresh set of resultants, and so on, until it is found that every resultant after a certain number of trials coincides with the corresponding resultant of the preceding trial. The diagrams of Fig. 53 represent the result of the sixth trial, when the variation between every resultant of displaced forces and the corresponding resultant belonging to the fifth trial was found to be negligible.

TABLE 8.—MÜLLER-BRESLAU METHOD. EQUATIONS INVOLVING ρ_1 TO ρ_{42}

$\Sigma M = 0$ gives:

$$
\left\{
\begin{aligned}
&5.464\,\rho_3 + 2.501\,\rho_2 = 0 \ldots\ldots (1)\\
&7.951\,\rho_{11} + 1.685\,\rho_{10} + 0.303\,\rho_5 + 5.464\,\rho_4 = 0 \ldots\ldots (2)\\
&2.501\,\rho_1 + 0.303\,\rho_6 + 2.501\,\rho_7 = 0 \ldots\ldots (3)\\
&7.951\,\rho_{18} + 0.190\,\rho_{13} + 7.951\,\rho_{12} = 0 \ldots\ldots (4)\\
&2.501\,\rho_8 + 1.685\,\rho_9 + 0.190\,\rho_{14} + 1.800\,\rho_{15} + 3.588\,\rho_{19} = 0 \ldots\ldots (5)\\
&7.951\,\rho_{27} + 1.800\,\rho_{26} + 0.303\,\rho_{22} + 1.800\,\rho_{16} + 7.951\,\rho_{17} = 0 \ldots\ldots (6)\\
&3.588\,\rho_{20} + 0.303\,\rho_{21} + 3.588\,\rho_{23} = 0 \ldots\ldots (7)\\
&7.951\,\rho_{34} + 0.190\,\rho_{29} + 7.951\,\rho_{28} = 0 \ldots\ldots (8)\\
&3.588\,\rho_{24} + 1.800\,\rho_{25} + 0.190\,\rho_{30} + 1.685\,\rho_{31} + 2.501\,\rho_{35} = 0 \ldots\ldots (9)\\
&5.464\,\rho_{39} + 0.303\,\rho_{38} + 1.685\,\rho_{32} + 7.951\,\rho_{33} = 0 \ldots\ldots (10)\\
&2.501\,\rho_{36} + 0.303\,\rho_{37} + 2.501\,\rho_{42} = 0 \ldots\ldots (11)\\
&2.501\,\rho_{41} + 5.464\,\rho_{40} = 0 \ldots\ldots (12)
\end{aligned}
\right.
$$

Joint (1):
$$
\begin{aligned}
&5.464\,\rho_3 + 2.501\,\rho_2 = 0\\
&\text{or,}\quad \rho_3 = -0.4577\,\rho_2.
\end{aligned}
$$

Triangle A:
$$
\left\{
\begin{aligned}
(2\,\rho_3 - \rho_4) &= +\ 96.66 + 2\,\rho_2 - \rho_1\\
\rho_4 &= -\ 96.66 - 2\,\rho_2 + \rho_1 + 2\,\rho_3\\
&= -\ 96.66 + \rho_1 - 2.9154\,\rho_2\\
(\rho_5 - \rho_4) &= +136.56 + (\rho_1 - \rho_2)\\
\rho_5 &= +136.56 + \rho_1 - \rho_2 + \rho_4\\
&= +\ 39.90 + 2\,\rho_1 - 3.9154\,\rho_2\\
(\rho_1 - \rho_6) &= -233.22 + \rho_3 - \rho_4\\
\rho_6 &= +233.22 - \rho_3 + \rho_4 + \rho_1\\
&= +136.56 + 2\,\rho_1 - 2.4577\,\rho_2
\end{aligned}
\right.
$$

Joint (3):
$$
\begin{aligned}
&2.501\,\rho_1 + 0.303\,\rho_6 + 2.501\,\rho_7 = 0\\
&\therefore \rho_7 = -\rho_1 - 0.1212\,\rho_6\\
& = -16.55 - 1.2424\,\rho_1 + 0.2979\,\rho_2
\end{aligned}
$$

Triangle B:
$$
\left\{
\begin{aligned}
2\,\rho_6 - \rho_5 &= +194.34 + 2\,\rho_7 - \rho_8\\
\rho_8 &= +194.34 + 2\,\rho_7 - 2\,\rho_6 + \rho_5\\
&= -\ 71.98 - 4.4848\,\rho_1 + 1.5958\,\rho_2\\
\rho_8 - \rho_9 &= -162.42 + \rho_6 - \rho_5\\
\rho_9 &= +162.42 - \rho_6 + \rho_5 + \rho_8\\
&= -\ 6.22 - 4.4848\,\rho_1 + 0.1381\,\rho_2\\
\rho_{10} - \rho_5 &= -\ 31.92 + \rho_8 - \rho_7\\
\rho_{10} &= -\ 31.92 + \rho_8 - \rho_7 + \rho_5\\
&= -\ 47.45 - 1.2424\,\rho_1 - 2.6175\,\rho_2
\end{aligned}
\right.
$$

Joint (2):
$$
\begin{aligned}
&7.951\,\rho_{11} + 1.685\,\rho_{10} + 0.303\,\rho_5 + 5.464\,\rho_4 = 0\\
&\therefore \rho_{11} = -0.2119\,\rho_{10} - 0.0381\,\rho_5 - 0.6872\,\rho_4\\
&\phantom{\therefore \rho_{11}} = +74.93 - 0.5001\,\rho_1 + 2.7073\,\rho_2
\end{aligned}
$$

Triangle C:
$$
\left\{
\begin{aligned}
2\,\rho_{11} - \rho_{12} &= -162.42 + 2\,\rho_{10} - \rho_9\\
\rho_{12} &= +162.42 - 2\,\rho_{10} + \rho_9 + 2\,\rho_{11}\\
&= +400.96 - 3.0002\,\rho_1 + 10.7877\,\rho_2\\
\rho_{13} - \rho_{12} &= +358.08 + \rho_9 - \rho_{10}\\
\rho_{13} &= +358.08 + \rho_9 - \rho_{10} + \rho_{12}\\
&= +800.27 - 6.2426\,\rho_1 + 13.5483\,\rho_2\\
\rho_9 - \rho_{14} &= -195.66 + \rho_{11} - \rho_{12}\\
\rho_{14} &= +195.66 - \rho_{11} - \rho_{12} + \rho_9\\
&= +515.47 - 6.9849\,\rho_1 + 8.2185\,\rho_2
\end{aligned}
\right.
$$

Joint (4):
$$
\begin{aligned}
&7.951\,\rho_{18} + 0.190\,\rho_{13} + 7.951\,\rho_{12} = 0\\
&\therefore \rho_{18} = -0.0239\,\rho_{13} - \rho_{12}\\
&\phantom{\therefore \rho_{18}} = -420.09 + 3.1494\,\rho_1 - 10.9841\,\rho_2
\end{aligned}
$$

Triangle D:
$$
\left\{
\begin{aligned}
2\,\rho_{18} - \rho_{17} &= -153.66 + 2\,\rho_{13} - \rho_{14}\\
\rho_{17} &= +153.66 - 2\,\rho_{13} - \rho_{14} + 2\,\rho_{18}\\
&= -1\,771.59 + 11.7991\,\rho_1 - 40.8363\,\rho_2\\
\rho_{16} - \rho_{17} &= +147.66 + \rho_{14} - \rho_{13}\\
\rho_{16} &= +147.66 + \rho_{14} - \rho_{13} + \rho_{17}\\
&= -1\,908.73 + 11.0568\,\rho_1 - 46.1611\,\rho_2\\
\rho_{14} - \rho_{15} &= +6.00 + \rho_{18} - \rho_{17}\\
\rho_{15} &= -6.00 - \rho_{18} + \rho_{17} + \rho_{14}\\
&= -842.03 + 1.6488\,\rho_1 - 21.6387\,\rho_2
\end{aligned}
\right.
$$

Joint (5):
$$
\begin{aligned}
&2.501\,\rho_8 + 1.685\,\rho_9 + 0.190\,\rho_{14} + 1.800\,\rho_{15} + 3.588\,\rho_{19} = 0\\
&\therefore \rho_{19} = -0.6970\,\rho_8 - 0.4696\,\rho_9 - 0.0530\,\rho_{14} - 0.5017\,\rho_{15}\\
&\phantom{\therefore \rho_{19}} = +448.22 + 4.7670\,\rho_1 + 9.2408\,\rho_2
\end{aligned}
$$

Triangle E:
$$
\left\{
\begin{aligned}
2\,\rho_{15} - \rho_{16} &= +147.66 + 2\,\rho_{19} - \rho_{20}\\
\rho_{20} &= +147.66 + 2\,\rho_{19} + \rho_{16} - 2\,\rho_{15}\\
&= +819.43 + 17.2612\,\rho_1 + 15.5879\,\rho_2\\
\rho_{20} - \rho_{21} &= -359.94 + \rho_{15} - \rho_{16}\\
\rho_{21} &= +359.94 - \rho_{15} + \rho_{16} + \rho_{20}\\
&= +112.67 + 26.6532\,\rho_1 - 8.9395\,\rho_2\\
\rho_{22} - \rho_{16} &= +212.28 + \rho_{20} - \rho_{19}\\
\rho_{22} &= +212.28 + \rho_{20} - \rho_{19} + \rho_{16}\\
&= -1\,325.24 + 23.5510\,\rho_1 - 39.8140\,\rho_2
\end{aligned}
\right.
$$

TABLE 8.—(*Continued*).

Joint (7):

$$
\begin{aligned}
&3.588\,\rho_{20} + 0.303\,\rho_{21} + 3.588\,\rho_{23} = 0\\
&\rho_{23} = -\rho_{20} - 0.0844\,\rho_{21}\\
&\quad = -828.94 - 19.5107\,\rho_1 - 14.8334\,\rho_2
\end{aligned}
$$

Triangle F:

$$
\begin{cases}
2\,\rho_{21} - \rho_{22} = +127.50 + 2\,\rho_{23} - \rho_{24}\\
\rho_{24} = +127.50 + 2\,\rho_{23} + \rho_{22} - 2\,\rho_{21}\\
\quad = -3\,080.96 - 68.7768\,\rho_1 - 51.6018\,\rho_2\\
\rho_{24} - \rho_{25} = -147.66 + \rho_{21} - \rho_{22}\\
\rho_{25} = +147.66 - \rho_{21} + \rho_{22} + \rho_{24}\\
\quad = -4\,371.21 - 71.8790\,\rho_1 - 82.4763\,\rho_2\\
\rho_{26} - \rho_{22} = +20.16 + \rho_{24} - \rho_{23}\\
\rho_{26} = +20.16 + \rho_{24} - \rho_{23} + \rho_{22}\\
\quad = -3\,557.10 - 25.7151\,\rho_1 - 76.5824\,\rho_2
\end{cases}
$$

Joint (6):

$$
\begin{aligned}
&7.951\,\rho_{27} + 1.800\,\rho_{26} + 0.303\,\rho_{22} + 1.800\,\rho_{16} + 7.951\,\rho_{17} = 0\\
&\rho_{27} = -0.2264\,\rho_{26} - 0.0381\,\rho_{22} - 0.2264\,\rho_{16} - \rho_{17}\\
&\quad = +3\,059.55 - 9.3778\,\rho_1 + 70.1424\,\rho_2
\end{aligned}
$$

Triangle G:

$$
\begin{cases}
2\,\rho_{27} - \rho_{28} = -147.66 + 2\,\rho_{26} - \rho_{25}\\
\rho_{28} = +147.66 - 2\,\rho_{26} + \rho_{25} + 2\,\rho_{27}\\
\quad = +9\,009.75 - 39.2044\,\rho_1 + 210.9733\,\rho_2\\
\rho_{29} - \rho_{28} = +423.84 + \rho_{25} - \rho_{26}\\
\rho_{29} = +423.84 + \rho_{28} + \rho_{25} - \rho_{26}\\
\quad = +8\,619.48 - 85.3683\,\rho_1 + 205.0794\,\rho_2\\
\rho_{25} - \rho_{30} = -276.18 + \rho_{27} - \rho_{28}\\
\rho_{30} = +276.18 - \rho_{27} + \rho_{28} + \rho_{25}\\
\quad = +1\,855.17 - 101.7056\,\rho_1 + 58.3546\,\rho_2
\end{cases}
$$

Joint (8):

$$
\begin{aligned}
&7.951\,\rho_{34} + 0.190\,\rho_{29} + 7.951\,\rho_{28} = 0\\
&\rho_{34} = -0.0239\,\rho_{29} - \rho_{28}\\
&\quad = -9\,215.76 + 41.2447\,\rho_1 - 215.8747\,\rho_2
\end{aligned}
$$

Triangle H:

$$
\begin{cases}
2\,\rho_{34} - \rho_{36} = +716.34 + 2\,\rho_{29} - \rho_{30}\\
\rho_{33} = -716.34 - 2\,\rho_{29} + \rho_{30} + 2\,\rho_{34}\\
\quad = -34\,531.65 + 151.5204\,\rho_1 - 783.5536\,\rho_2\\
\rho_{32} - \rho_{33} = -324.90 + \rho_{30} - \rho_{29}\\
\rho_{32} = -324.90 + \rho_{33} + \rho_{30} - \rho_{29}\\
\quad = -41\,620.86 + 136.1831\,\rho_1 - 930.2784\,\rho_2\\
\rho_{30} - \rho_{31} = -391.44 + \rho_{34} - \rho_{33}\\
\rho_{31} = +391.44 - \rho_{34} + \rho_{33} + \rho_{30}\\
\quad = -23\,069.28 + 8.5701\,\rho_1 - 509.3243\,\rho_2
\end{cases}
$$

Joint (9):

$$
\begin{aligned}
&3.588\,\rho_{24} + 1.800\,\rho_{25} + 0.190\,\rho_{30} + 1.685\,\rho_{31} + 2.501\,\rho_{35} = 0\\
&\rho_{35} = -1.4346\,\rho_{24} - 0.7197\,\rho_{25} - 0.0760\,\rho_{30} - 0.6737\,\rho_{31}\\
&\quad = +22\,966.69 + 152.3544\,\rho_1 + 472.0830\,\rho_2
\end{aligned}
$$

Triangle I:

$$
\begin{cases}
2\,\rho_{31} - \rho_{32} = -324.90 + 2\,\rho_{35} - \rho_{36}\\
\rho_{36} = -324.90 + 2\,\rho_{35} + \rho_{32} - \rho_{31}\\
\quad = +50\,126.18 + 423.7517\,\rho_1 + 1\,032.5362\,\rho_2\\
\rho_{38} - \rho_{32} = -63.90 + \rho_{36} - \rho_{35}\\
\rho_{38} = -63.90 + \rho_{36} - \rho_{35} + \rho_{32}\\
\quad = -14\,525.27 + 407.5804\,\rho_1 - 369.8252\,\rho_2\\
\rho_{36} - \rho_{37} = +388.80 + \rho_{31} - \rho_{32}\\
\rho_{37} = -388.80 - \rho_{31} + \rho_{32} + \rho_{36}\\
\quad = +31\,185.80 + 551.3647\,\rho_1 + 611.5821\,\rho_2
\end{cases}
$$

Joint (11):

$$
\begin{aligned}
&2.501\,\rho_{36} + 0.303\,\rho_{37} + 2.501\,\rho_{42} = 0\\
&\rho_{42} = -\rho_{36} - 0.1212\,\rho_{37}\\
&\quad = -53\,905.90 - 490.5771\,\rho_1 - 1\,106.6600\,\rho_2
\end{aligned}
$$

Triangle J:

$$
\begin{cases}
2\,\rho_{37} - \rho_{38} = -466.44 + 2\,\rho_{42} - \rho_{41}\\
\rho_{41} = -466.44 + 2\,\rho_{42} + \rho_{38} - 2\,\rho_{37}\\
\quad = -185\,175.11 - 1\,676.3032\,\rho_1 - 3\,806.3094\,\rho_2\\
\rho_{39} + \rho_{38} = +273.12 + \rho_{41} + \rho_{42}\\
\rho_{39} = +273.12 + \rho_{41} + \rho_{42} + \rho_{38}\\
\quad = -145\,521.36 - 777.7228\,\rho_1 - 3\,069.4746\,\rho_2\\
\rho_{41} - \rho_{40} = +193.32 + \rho_{37} + \rho_{38}\\
\rho_{40} = -193.32 - \rho_{37} + \rho_{38} + \rho_{41}\\
\quad = -231\,079.50 - 1\,820.0875\,\rho_1 - 4\,787.7167\,\rho_2
\end{cases}
$$

Joints (10) and (12) give the two simultaneous equations:

$$
\begin{aligned}
5.464\,\rho_{39} + 0.303\,\rho_{38} + 1.685\,\rho_{32} + 7.951\,\rho_{33} &= 0\\
2.501\,\rho_{41} + 5.464\,\rho_{40} &= 0
\end{aligned}
$$

In terms of ρ_1 and ρ_2, these become:

$$
\begin{aligned}
-1\,144\,222.17 - 2\,691.7733\,\rho_1 - 24\,681.2200\,\rho_2 &= 0,\\
-1\,725\,741.34 - 14\,137.3924\,\rho_1 - 35\,679.6638\,\rho_2 &= 0.
\end{aligned}
$$

that is,

$$
\begin{aligned}
+425.081 + \rho_1 + 9.16913\,\rho_2 &= 0\\
+122.069 + \rho_1 + 2.52378\,\rho_2 &= 0.\\
\therefore +303.012 \qquad\quad + 6.64535\,\rho_2 &= 0\\
\therefore \rho_2 &= -45.5976\\
\rho_1 &= -6.9907
\end{aligned}
$$

Substituting these values in the ρ-expressions, Table 9 is obtained.

TABLE 9.—MÜLLER-BRESLAU METHOD. SOLUTION OF VALUES OF ρ_3 TO ρ_{42}.

$$\rho_3 = +20.8700$$
$$\rho_4 = -96.66 - 6.9907 + 132.9352 = +29.2845$$
$$\rho_5 = +39.90 - 13.9814 + 178.5328 = +204.4514$$
$$\rho_6 = +136.56 - 13.9814 + 112.0652 = +234.6438$$
$$\rho_7 = -16.55 + 8.6852 - 13.5835 = -21.4483$$
$$\rho_8 = -71.98 + 31.3519 - 72.7647 = -113.3928$$
$$\rho_9 = -6.22 + 31.3519 - 6.2970 = +18.8340$$
$$\rho_{10} = -47.45 + 8.6852 + 119.3517 = +80.5869$$
$$\rho_{11} = +74.93 + 3.4960 - 123.4464 = -45.0204$$
$$\rho_{12} = +400.96 + 20.9735 - 491.8932 = -69.9597$$
$$\rho_{13} = +800.27 + 43.6401 - 617.5420 = +226.3681$$
$$\rho_{14} = +515.47 + 48.8293 - 374.7439 = +189.5554$$
$$\rho_{15} = -842.03 - 11.6381 + 986.4448 = +132.7767$$
$$\rho_{16} = -1\ 908.73 - 77.2948 + 2\ 104.8354 = +118.8106$$
$$\rho_{17} = -1\ 771.59 - 82.4840 + 1\ 862.0373 = +7.9633$$
$$\rho_{18} = -420.09 - 22.0165 + 500.8486 = +58.7421$$
$$\rho_{19} = +448.22 - 33.3247 - 421.3583 = -6.4630$$
$$\rho_{20} = +819.43 - 120.6679 - 710.7708 = -12.0087$$
$$\rho_{21} = +112.67 - 186.3245 + 407.6197 = +333.9652$$
$$\rho_{22} = -1\ 325.24 - 164.6380 + 1\ 815.4228 = +325.5448$$
$$\rho_{23} = -828.94 + 136.3935 + 676.3674 = -16.1791$$
$$\rho_{24} = -3\ 080.96 + 480.7980 + 2\ 352.9182 = -247.2438$$
$$\rho_{25} = -4\ 371.21 + 502.4845 + 3\ 760.7213 = -108.0042$$
$$\rho_{26} = -3\ 557.10 + 179.7665 + 3\ 491.9736 = +114.6401$$
$$\rho_{27} = +3\ 059.55 + 65.5574 - 3\ 198.3251 = -73.2177$$
$$\rho_{28} = +9\ 009.75 + 274.0662 - 9\ 619.8761 = -336.0599$$
$$\rho_{29} = +8\ 619.48 + 596.7842 - 9\ 351.1284 = -134.8642$$
$$\rho_{30} = +1\ 855.17 + 710.9933 + 2\ 660.8297 = -94.6664$$
$$\rho_{31} = -23\ 069.28 - 59.9110 + 23\ 223.9657 = +94.7747$$
$$\rho_{32} = -41\ 620.86 - 952.0152 + 42\ 418.4624 = -154.4128$$
$$\rho_{33} = -34\ 531.65 - 1\ 059.2337 + 35\ 728.1636 = +137.2799$$
$$\rho_{34} = -9\ 215.76 - 288.3293 + 9\ 843.3682 = +339.2789$$
$$\rho_{35} = +22\ 966.69 - 1\ 065.0639 - 21\ 525.8518 = +375.7743$$
$$\rho_{36} = +50\ 126.18 - 2\ 962.3210 - 4\ 7081.1726 = +82.6864$$
$$\rho_{37} = +31\ 185.80 - 3\ 854.4252 - 27\ 886.6760 = -555.3012$$
$$\rho_{38} = -14\ 525.27 - 2\ 849.2723 + 16\ 863.1415 = +511.4008$$
$$\rho_{39} = -145\ 521.36 + 5\ 436.8268 + 139\ 960.6750 = -123.8582$$
$$\rho_{40} = -231\ 079.50 + 12\ 723.6857 + 218\ 308.3910 = +47.4233$$
$$\rho_{41} = -185\ 175.11 + 11\ 718.5328 + 173\ 558.5735 = +101.9963$$
$$\rho_{42} = -53\ 905.90 + 3\ 429.4773 + 50\ 461.0400 = -15.3827$$

TABLE 10.—MÜLLER-BRESLAU METHOD. SECONDARY STRESSES IN MEMBERS.

Joint.	Member.	Extreme fiber distance, c, in inches.	l, in inches.	$\frac{2c}{l}$.	ρ.	$f = \frac{c}{l} \times \rho$ in pounds per square inch.
1	1-2	9.38 9.49	477.9	0.03925 0.03971	+ 20.87	− 0.41 + 0.41
	1-3	7.5	300	0.05	− 45.60	± 1.14
2	2-4	9.33 9.54	300	0.0622 0.0636	− 45.03	+ 1.40 − 1.43
	2-5	7.5	477.9	0.03139	+ 80.59	∓ 1.26
	2-3	5.28	372	0.02839	+204.45	∓ 2.90
	2-1	9.49 9.38	477.9	0.03971 0.03925	+ 29.28	− 0.58 + 0.57
3	3-1	7.5	300	0.05	− 6.99	± 0.18
	3-2	5.28	372	0.02839	+234.64	∓ 3.33
	3-5	7.5	300	0.05	− 21.45	± 0.54
4	4-6	9.33 9.54	300	0.0622 0.0636	+ 58.74	− 1.83 + 1.87
	4-5	5.19	372	0.02790	+226.37	∓ 3.16
	4-2	9.54 9.33	300	0.0636 0.0622	− 69.96	+ 2.22 − 2.17
5	5-3	7.5	300	0.05	−113.39	± 2.83
	5-2	7.5	477.9	0.03139	+ 18.83	∓ 0.30
	5-4	5.19	372	0.02790	+189.56	∓ 2.64
	5-6	7.5	477.9	0.03139	+132.78	∓ 2.08
	5-7	7.5	300	0.05	− 6.46	± 0.16
6	6-8	9.33 9.54	300	0.0622 0.0636	− 73.22	+ 2.28 − 2·33
	6-9	7.5	477.9	0.03139	+114·64	∓ 1.80
	6-7	5.28	372	0.02839	+325.54	∓ 4.62
	6-5	7.5	477.9	0.03139	+118.81	∓ 1.86
	6-4	9.54 9.33	300	0.0636 0.0622	+ 7.96	− 0.25 + 0.25
7	7-5	7.5	300	0.05	− 12.01	± 0.30
	7-6	5.28	372	0.02839	+333.97	∓ 4.74
	7-9	7.5	300	0.05	− 16.18	± 0.40
8	8-10	9.33 9.54	300	0.0622 0.0636	+339.28	−10.55 +10.79
	8-9	5.19	372	0.02790	−134.86	± 1.88
	8-6	9.54 9.33	300	0.0636 0.0622	−336.06	+10.69 −10.45
9	9-7	7.5	300	0.05	−247.24	± 6.18
	9-6	7.5	477.9	0.03139	−108.00	± 1.69
	9-8	5.19	372	0.02790	− 94.67	± 1.32
	9-10	7.5	477.9	0.03139	+ 94.77	∓ 1.49
	9-11	7.5	300	0.05	+375.77	∓ 9.39
10	10-12	9.38 9.49	477.9	0.03925 0.03971	−123.86	+ 2.43 − 2.44
	10-11	5·28	372	0.02839	−511.40	± 7.26
	10-9	7.5	477.9	0.03139	−154.41	± 2.42
	10-8	9.54 9.33	300	0.0636 0.0622	+137.28	− 4.36 + 4.27
11	11-9	7.5	300	0.05	+ 82.69	∓ 2.07
	11-10	5.28	372	0.02839	−555.30	± 7.88
	11-12	7.5	300	0.05	− 15.38	± 0.38
12	12-11	7.5	300	0.05	+101.99	∓ 2.55
	12-10	9.49 9.38	477.9	0.03971 0.03925	− 47.42	+ 0.94 − 0.93

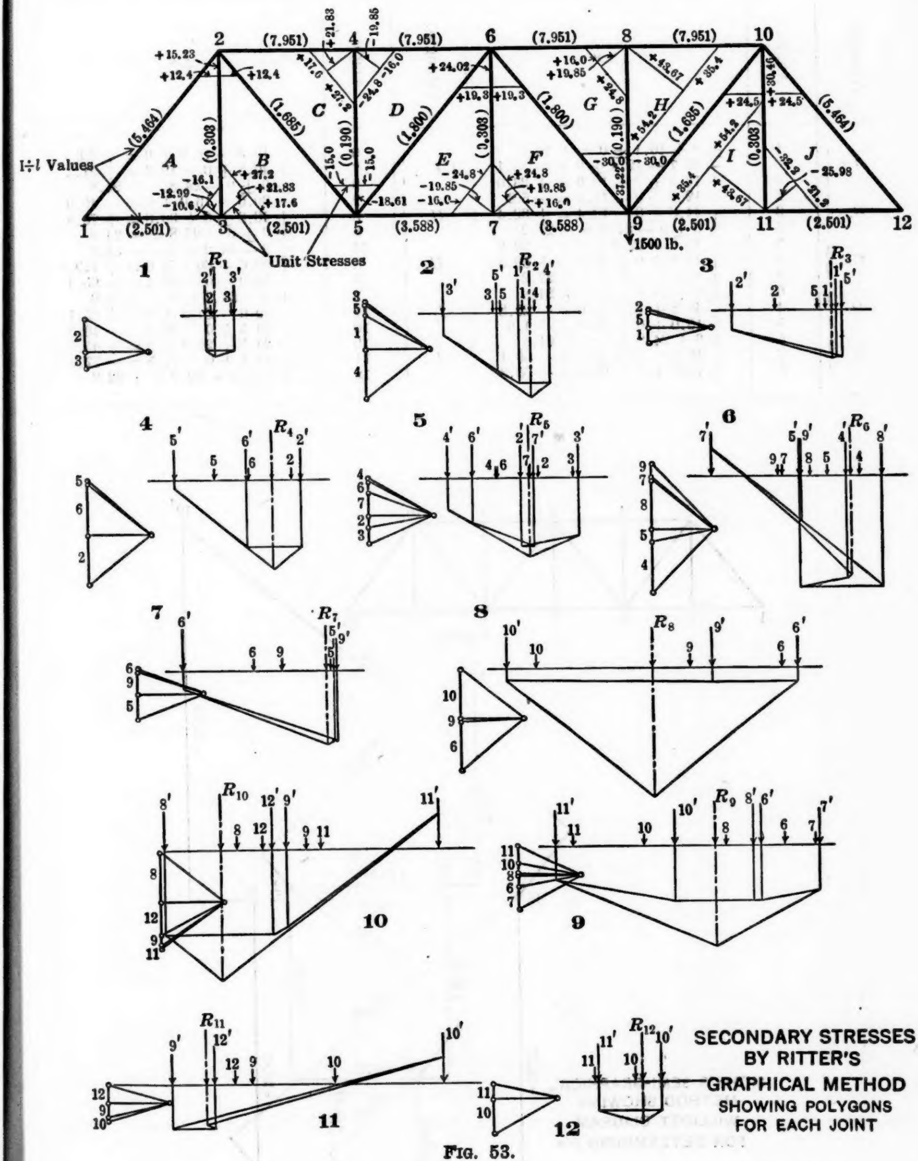
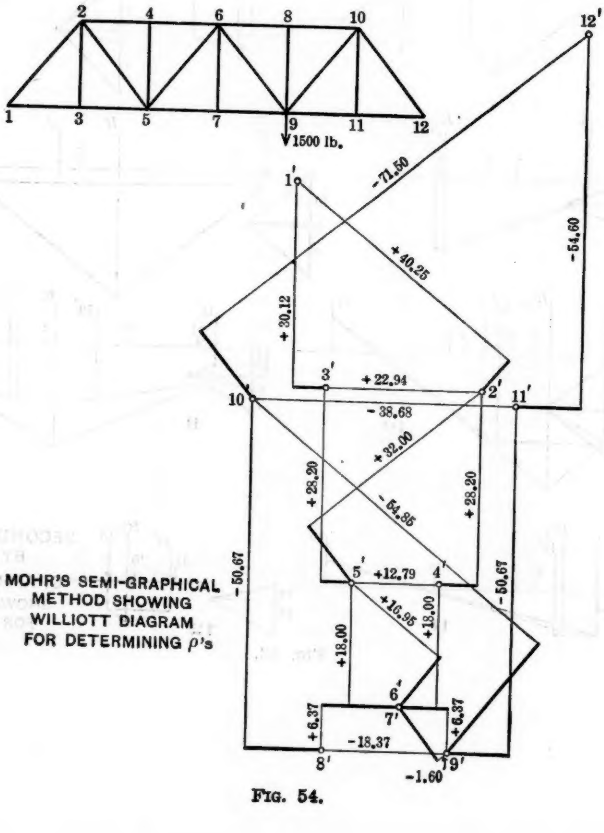

I÷l Values
Unit Stresses
1500 lb.
SECONDARY STRESSES
BY RITTER'S
GRAPHICAL METHOD
SHOWING POLYGONS
FOR EACH JOINT

FIG. 53.

TABLE 11.—RITTER'S GRAPHICAL METHOD. VALUES OF $\Delta\theta$.

Triangle.	Angle.	$\Delta\theta$.	Triangle.	Angle.	$\Delta\theta$.
	1	$\pm 0 + 16.1 \pm 0 = + 16.1$		6	$+ 19.3 - 16.0 \pm 0 = + 3.3$
A.	3	$- 26.7 - 12.4 \pm 0 = - 39.1$	*F.*	7	$+ 40.8 \pm 0 - 19.3 = + 21.5$
	2	$+ 12.4 \pm 0 + 10.6 = + 23.0$		9	$\pm 0 \pm 0 - 24.8 = - 24.8$
	2	$+ 12.4 \pm 0 - 17.6 = - 5.2$		9	$- 30.0 - 16.0 \pm 0 = - 46.0$
B.	3	$+ 44.8 \pm 0 - 12.4 = + 32.4$	*G.*	8	$+ 40.8 \pm 0 + 30.0 = + 70.8$
	5	$\pm 0 \pm 0 - 27.2 = - 27.2$		6	$\pm 0 \pm 0 - 24.8 = - 24.8$
	5	$- 15.0 - 17.6 \pm 0 = - 32.6$		10	$\pm 0 - 54.2 \pm 0 = - 54.2$
C.	4	$+ 44.8 \pm 0 + 15.0 = + 59.8$	*H.*	8	$+ 89.6 + 30.0 \pm 0 = + 119.6$
	2	$\pm 0 \pm 0 - 27.2 = - 27.2$		9	$- 30.0 \pm 0 - 35.4 = - 65.4$
	6	$\pm 0 + 24.8 \pm 0 = + 24.8$		9	$\pm 0 - 54.2 \pm 0 = - 54.2$
D.	4	$- 40.8 + 15.0 \pm 0 = - 25.8$	*I.*	11	$+ 89.6 - 24.5 \pm 0 = + 65.1$
	5	$- 15.0 \pm 0 + 16.0 = + 1.0$		10	$+ 24.5 \pm 0 - 35.4 = - 10.9$
	5	$\pm 0 + 24.8 \pm 0 = + 24.8$		10	$+ 24.5 + 21.2 \pm 0 = + 45.7$
E.	7	$- 40.8 - 19.3 \pm 0 = - 60.1$	*J.*	11	$- 53.4 \pm 0 - 24.5 = - 77.9$
	6	$+ 19.3 \pm 0 + 16.0 = + 35.3$		12	$\pm 0 \pm 0 + 32.2 = + 32.2$



MOHR'S SEMI-GRAPHICAL METHOD SHOWING WILLIOTT DIAGRAM FOR DETERMINING ρ's

FIG. 54.

TABLE 12.—RITTER'S GRAPHICAL METHOD. SECONDARY STRESSES IN MEMBERS.

Joint.	2μ.	C.	l.	$\frac{2C}{l}$	$f = \frac{6C}{l}\mu$, in pounds per square inch.
1	$2\mu_2 = +7.0$	9.38	477.9	0.03926	-0.41
		9.49		0.03972	$+0.42$
	$2\mu_3 = -15.5$	7.5	300	0.05	± 1.16
2	$2\mu_1 = +9.6$	9.49	477.9	0.03972	-0.57
		9.38		0.03926	$+0.57$
	$2\mu_3 = +68.0$	5.28	372	0.02839	∓ 2.90
	$2\mu_4 = -14.6$	9.33	300	0.0622	$+1.36$
		9.54		0.0636	-1.39
	$2\mu_5 = +26.6$	7.5	477.9	0.03139	∓ 1.25
3	$2\mu_1 = -3.3$	7.5	300	0.05	± 0.25
	$2\mu_2 = +78.0$	5.28	372	0.02839	∓ 3.32
	$2\mu_5 = -7.8$	7.5	300	0.05	± 0.58
4	$2\mu_2 = -22.5$	9.54	300	0.0636	$+2.15$
		9.33		0.0622	-2.10
	$2\mu_5 = +76.6$	5.19	372	0.02790	∓ 3.21
	$2\mu_6 = +20.2$	9.33	300	0.622	-1.88
		9.54		0.636	$+1.93$
5	$2\mu_2 = +5.9$	7.5	477.9	0.03139	∓ 0.28
	$2\mu_3 = -38.5$	7.5	300	0.05	± 2.89
	$2\mu_4 = +63.6$	5.19	372	0.02790	∓ 2.66
	$2\mu_6 = +44.0$	7.5	477.9	0.03139	∓ 2.07
	$2\mu_7 = -2.9$	7.5	300	0.05	± 0.22
6	$2\mu_4 = +2.6$	9.54	300	0.0636	-0.25
		9.33		0.0622	$+0.24$
	$2\mu_5 = +39.3$	7.5	477.9	0.03139	∓ 1.85
	$2\mu_7 = +108.5$	5.28	372	0.02839	∓ 4.62
	$2\mu_8 = -24.8$	9.33	300	0.0622	$+2.31$
		9.54		0.0636	-2.37
	$2\mu_9 = +38.1$	7.5	477.9	0.03139	∓ 1.79
7	$2\mu_5 = -4.3$	7.5	300	0.05	± 0.32
	$2\mu_6 = +111.4$	5.28	372	0.02839	∓ 4.74
	$2\mu_9 = -5.8$	7.5	300	0.05	± 0.43
8	$2\mu_6 = -112.1$	9.54	300	0.0636	$+10.64$
		9.33		0.0622	-10.41
	$2\mu_9 = -44.0$	5.19	372	0.02790	± 1.84
	$2\mu_{10} = +113.1$	9.33	300	0.0622	-10.55
		9.54		0.0636	$+10.79$
9	$2\mu_6 = -35.8$	7.5	477.9	0.03139	± 1.69
	$2\mu_7 = -82.5$	7.5	300	0.05	± 6.19
	$2\mu_8 = -30.8$	5.19	372	0.02790	± 1.29
	$2\mu_{10} = +31.1$	7.5	477.9	0.03139	∓ 1.46
	$2\mu_{11} = +124.4$	7.5	300	0.05	∓ 9.33
10	$2\mu_8 = +44.4$	9.54	300	0.0636	-4.24
		9.33		0.0622	$+4.14$
	$2\mu_9 = -50.9$	7.5	477.9	0.03139	± 2.40
	$2\mu_{11} = -169.9$	5.28	372	0.02839	± 7.23
	$2\mu_{12} = -39.7$	9.38	477.9	0.03926	$+2.34$
		9.49		0.03972	-2.37
11	$2\mu_9 = +27.2$	7.5	300	0.05	∓ 2.04
	$2\mu_{10} = -185.2$	5.28	372	0.02839	± 7.89
	$2\mu_{12} = -5.4$	7.5	300	0.05	± 0.40
12	$2\mu_{10} = -15.5$	9.49	477.9	0.03972	$+0.92$
		9.38		0.03926	-0.91
	$2\mu_{11} = +34.0$	7.5	300	0.05	∓ 2.55

The quantities, 2 μ, were then listed in Table 12. For example, in the figure for Joint (1), 2′ lies at a distance of 0.7 to the left of R_1, the resultant of the displaced forces. Hence, 2 $\mu_2 = + 7.0$ and this is tabulated. Equation (54) is then used to determine the secondary stresses. The moments were to be positive when acting in a clockwise direction. The signs of the secondary stresses, therefore, are justified.

Mohr's Semi-Graphical Method.—The result of applying Mohr's semi-graphical method to the solution of the problem is shown in Fig. 54 and Tables 13 to 16, inclusive. Table 13 is partly filled (the first four columns), when the Williot diagram is drawn. The Δ l's are laid out to a convenient scale; this is on the supposition that E is unity. From the completed diagram are then obtained the ρ's to be entered in Column (5) of Table 13, which table can then be completed.

Table 14 is then constructed, by the use of which the equations for the joints may be built (see Equation (59)). Table 15 is the solution of these equations by Gauss' method of normal equations.

Table 16 is then completed, making use of Equation (60). Note, again, that a clockwise bending moment is reckoned to be positive in this analysis. Hence, it is that the + 0.105 of Member 1-2 gives a compressive stress for the top fiber and a tensile stress for the lower fiber of Member 1-2, End (1).

Mohr's Elastic Weight Method.—The solution of the problem by means of Mohr's elastic weight method is given in Fig. 55 and Tables 17 to 22, inclusive. Table 17 is filled out first. Then, the truss is separated into triangles (Fig. 55), and on each is indicated by arrows the direction of following around the chain. Thus, it is a clockwise rotation for Δ *A* followed by a counter-clockwise rotation for Δ *B*, etc. At the vertices of each triangle, place the ψ's belonging to that triangle. Thus, in going around Joint (2) of Δ *A*, the direction is from Member 1-2 to Member 2-3; ψ_2 is, therefore, $\lambda_{23} - \lambda_{12} = 0 - (-12.99) = + 12.99$.

Then, place the elastic weights at the third points of the members. Thus, $\phi_{12} = + 0.091\, M_{12}$ at the first "third" point on the way from Joint (1) to Joint (2), and $\phi_{21} = - 0.091 M_{21}$ at the second, etc.

Assume that M_{12} and M_{21} are known. The $\Sigma\, M = 0$ for Joint (1) gives Equation I-1 of Table 18. Taking moments of the ϕ's about the u_1-axis in Δ *A* and also moments of the ψ's about the v_1-axis, marked in Δ *A*, as the u_1-axis coincides with the v_1-axis on a clockwise rotation of 90°, the sum of the ϕ-moments, together with the sum of the ψ-moments, must be zero. (Refer to Equations (72)). In taking these axes, only one fresh unknown has been introduced, namely, M_{31}; Equation II-1 of Table 18 is the result. By taking moments about the axes, u_2 and v_2, it will be found that Equation II-2 contains only one fresh unknown, namely, M_{23}. Equation II-3 is the result of taking moments about u_3 and v_3.

Then Equation I-3 is the equation representing the fact that $\Sigma\, M_3 = 0$. The remainder of the equations in Table 18 follow in the same manner. The axes are marked 1, 2, 3, in the order and place, to make sure that only one new unknown is added with every additional moment equation. The height

TABLE 13.—MOHR'S SEMI-GRAPHICAL METHOD. ELEMENTS OF MEMBERS.

Member.	Length, l, in inches.	Unit stress, s, in pounds.	$\Delta l = \frac{s.\,l}{1\,000}$	ρ, in kips per square inch.	$\psi = \frac{\rho}{l}$, in 100 lb. per sq. in.	k, in cubic inches.	$k\,\psi$, in 100 in-lb.
1- 2	477.9	−12.99	− 6.208	+40.25	+0.842	5.464	+ 4.601
2- 4	300	−18.61	− 5.583	+28.20	+0.940	7.951	+ 7.474
4- 6	300	−18.61	− 5.583	+18.00	+0.600	7.951	+ 4.771
6- 8	300	−37.22	−11.166	+ 6.37	+0.212	7.951	+ 1.686
8-10	300	−37.22	−11.166	−50.67	−1.689	7.951	−13.429
10-12	477.9	−25.98	−12.416	−71.50	−1.496	5.464	− 8.174
1- 3	300	+15.23	+ 4.569	+30.12	+1.004	2.501	+ 2.511
3- 5	300	+15.23	+ 4.569	+28.20	+0.940	2.501	+ 2.351
5- 7	300	+24.02	+ 7.206	+18.00	+0.600	3.588	+ 2.153
7- 9	300	+24.02	+ 7.206	+ 6.37	+0.212	3.588	+ 0.761
9-11	300	+30.46	+ 9.138	−50.67	−1.689	2.501	− 4.224
11-12	300	+30.46	+ 9.138	−54.60	−1.820	2.501	− 4.552
2- 3	372	....	....	+22.94	+0.617	0.303	+ 0.187
4- 5	372	....	....	+12.79	+0.344	0.190	+ 0.065
6- 7	372	....	....	± 0	....	0.303	......
8- 9	372	....	....	−18.37	−0.494	0.190	− 0.094
10-11	372	....	....	−38.68	−1.040	0.303	− 0.315
2- 5	477.9	+21.83	+10.433	+32.00	+0.670	1.685	+ 1.129
5- 6	477.9	−19.85	− 9.486	+16.95	+0.355	1.800	+ 0.639
6- 9	477.9	+19.85	+ 9.486	− 1.60	−0.033	1.800	− 0.059
9-10	477.9	+43.67	+20.870	−54.85	−1.148	1.685	− 1.934

TABLE 14.—MOHR'S SEMI-GRAPHICAL METHOD. VALUES OF k AND $k\,\psi$.

Joint.	Member.	k, in cubic inches.	$k\psi$, in 100 in.-lb.	Joint.	Member.	k, in cubic inches.	$k\psi$, in 100 in.-lb.
1	1-2	5.464	+ 4.601	7	7-5	3.588	+ 2.153
	1-3	2.501	+ 2.511		7-6	0.303	... .
	Σ	7.965	+ 7.112		7-9	3.588	+ 0.761
2	2-4	7.951	+ 7.474		Σ	7.479	+ 2.914
	2-5	1.685	+ 1.129	8	8-10	7.951	−13.429
	2-3	0.303	+ 0.187		8-9	0.190	− 0.094
	2-1	5.464	+ 4.601		8-6	7.951	+ 1.686
	Σ	15.403	+13.391		Σ	16.092	−11.837
3	3-1	2.501	+ 2.511	9	9-7	3.588	+ 0.761
	3-2	0.303	+ 0.187		9-6	1.800	− 0.059
	3-5	2.501	+ 2.351		9-8	0.190	− 0.094
	Σ	5.305	+ 5.049		9-10	1.685	− 1.934
4	4-6	7.951	+ 4.771		9-11	2.501	− 4.224
	4-5	0.190	+ 0.065		Σ	9.764	− 5.550
	4-2	7.951	+ 7.474	10	10-12	5.464	− 8.174
	Σ	16.092	+12.310		10-11	0.303	− 0.315
5	5-3	2.501	+ 2.351		10-9	1.685	− 1.934
	5-2	1.685	+ 1.129		10-8	7.951	−13.429
	5-4	0.190	+ 0.065		Σ	15.403	−23.852
	5-6	1.800	+ 0.639	11	11-9	2.501	− 4.224
	5-7	3.588	+ 2.153		11-10	0.303	− 0.315
	Σ	9.764	+ 6.337		11-12	2.501	− 4.552
6	6-8	7.951	+ 1.686		Σ	5.305	− 9.091
	6-9	1.800	− 0.059	12	12-11	2.501	− 4.552
	6-7	0.303	....		12-10	5.464	− 8.174
	6-5	1.800	+ 0.639		Σ	7.965	−12.726
	6-4	7.951	+ 4.771				
	Σ	19.805	+ 7.037				

TABLE 15.—Mohr's Semi-Graphical Method. Solution of Equations.

Number of equation.	Remarks.	ϕ_1.	ϕ_2.	ϕ_3.	ϕ_4.	ϕ_5.	ϕ_6.	ϕ_7.	ϕ_9.	ϕ_8.	ϕ_{11}.	ϕ_{10}.	ϕ_{12}.	Absolute term.	Check.
1	....	+15.930	+5.464	+2.501	....	....	....	....	....	....	....	....	....	−21.336	+2.559
1′	$1 \times -\frac{5.464}{15.930}$, *i. e.*, 0.34300	[−5.464]	−1.874	−0.858	....	....	....	....	....	....	....	....	....	+7.318	−0.878
2	....	[+5.464]	+30.806	+0.303	+7.951	+1.685	....	....	....	....	....	....	....	−40.173	+6.036
I	1′ + 2	....	+28.932	−0.555	+7.951	+1.685	....	....	....	....	....	....	....	−32.855	+5.158
1″	$1 \times -\frac{2.501}{15.930}$, *i. e.*, 0.15700	[−2.501]	[−0.858]	−0.393	....	....	....	....	....	....	....	....	....	+3.350	−0.402
I′	$\mathrm{I} \times +\frac{0.555}{28.932}$, *i. e.*, 0.01918	....	[+0.555]	−0.011	+0.153	+0.032	....	....	....	....	....	....	....	−0.630	+0.099
3	....	[+2.501]	[+0.303]	+10.610	....	+2.501	....	....	....	....	....	....	....	−15.147	+0.768
II	1″ + I′ + 3	....	....	+10.206	+0.153	+2.533	....	....	....	....	....	....	....	−12.427	+0.465
I″	$\mathrm{I} \times -\frac{7.951}{28.932}$, *i. e.*, 0.27482	....	[−7.951]	[+0.153]	−2.185	−0.463	....	....	....	....	....	....	....	+9.029	−1.418
II′	$\mathrm{II} \times -\frac{0.153}{10.206}$, *i. e.*, 0.01499	....	....	[−0.153]	−0.002	−0.038	....	....	....	....	....	....	....	+0.186	−0.007
4	....	....	[+7.951]	....	+32.184	+0.190	+7.951	....	....	....	....	....	....	−36.930	+11.346
III	I″ + II′ + 4	....	....	....	+29.997	−0.311	+7.951	....	....	....	....	....	....	−27.715	+9.921
I‴	$\mathrm{I} \times -\frac{1.685}{28.932}$, *i. e.*, 0.05824	....	[−1.685]	[+0.032]	[−0.463]	−0.098	....	....	....	....	....	....	....	+1.913	−0.300
II″	$\mathrm{II} \times -\frac{2.533}{10.206}$, *i. e.*, 0.24819	....	....	[−2.533]	[−0.038]	−0.629	....	....	....	....	....	....	....	+3.084	−0.115
III′	$\mathrm{III} \times +\frac{0.311}{9.997}$, *i. e.*, 0.01037	....	....	....	[+0.311]	−0.003	+0.082	....	....	....	....	....	....	−0.287	+0.103
5	....	....	[+1.685]	[+2.501]	[+0.190]	+19.528	+1.800	+3.588	....	....	....	....	....	−19.011	+10.281
IV	I‴ + II″ + III′ + 5	....	....	....	....	+18.798	+1.882	+3.588	....	....	....	....	....	−14.301	+9.969

TABLE 15.—(*Continued*).

Number of equation.	Remarks.	ϕ_1.	ϕ_2.	ϕ_3.	ϕ_4.	ϕ_5.	ϕ_6.	ϕ_7.	ϕ_9.	ϕ_8.	ϕ_{11}.	ϕ_{10}.	ϕ_{12}.	Absolute term.	Check.
III''	$\text{III} \times -\frac{7.951}{29.997}$, *i. e.*, 0.26506	......	......	......	[−7.951]	[+0.082]	−2.107	......	......	......	......	......	......	+7.346	−2.630
V'	$\text{IV} \times -\frac{1.882}{18.798}$, *i. e.*, 0.10012	......	......	......	......	[−1.882]	−0.188	−0.359	......	......	......	......	......	+1.432	−0.998
6	....	......	......	......	[+7.951]	[+1.800]	+39.610	+0.303	+1.800	+7.951	......	......	......	−21.111	+38.304
V	III'' + IV' + 6	......	......	......	......	......	+37.315	−0.056	+1.800	+7.951	......	......	......	−12.333	+34.676
IV''	$\text{IV} \times -\frac{3.588}{18.798}$, *i. e.*, 0.19087	......	......	......	......	[−3.588]	[−0.359]	−0.685	......	......	......	......	......	+2.730	−1.903
V'	$\text{V} \times +\frac{0.056}{37.315}$, *i. e.*, 0.00150	......	......	......	......	......	[+0.056]	−0.000	+0.003	+0.012	......	......	......	−0.019	+0.052
7	....	......	......	......	......	[+3.588]	[+0.303]	+14.958	+3.588	......	......	......	......	−8.742	+13.695
VI	IV'' + V' + 7	......	......	......	......	......	......	+14.273	+3.591	+0.012	......	......	......	−6.031	+11.844
V''	$\text{V} \times -\frac{1.800}{37.315}$, *i. e.*, 0.04824	......	......	......	......	......	[−1.800]	[+0.003]	−0.087	−0.384	......	......	......	+0.595	−1.673
VI'	$\text{VI} \times -\frac{3.591}{14.273}$, *i. e.*, 0.25159	......	......	......	......	......	......	[−3.591]	−0.903	−0.003	......	......	......	+1.517	−2.980
8	....	......	......	......	......	......	[+1.800]	[+3.588]	+19.528	+0.190	+2.501	+1.685	......	+16.650	+45.942
VII	V'' + VI' + 8	......	......	......	......	......	......	......	+18.538	−0.197	+2.501	+1.685	......	+18.762	+41.289
V'''	$\text{V} \times -\frac{7.951}{37.315}$, *i. e.*, 0.21308	......	......	......	......	......	[−7.951]	[+0.012]	[−0.384]	−1.694	......	......	......	+2.628	−7.386
VI''	$\text{VI} \times -\frac{0.012}{14.273}$, *i. e.*, 0.00084	......	......	......	......	......	......	[−0.012]	[−0.003]	−0.000	......	......	......	+0.005	−0.010
VII'	$\text{VII} \times +\frac{0.197}{18.538}$, *i. e.*, 0.01063	......	......	......	......	......	......	......	[+0.197]	−0.012	+0.027	+0.018	......	+0.199	+0.439
9	....	......	......	......	......	......	[+7.951]	......	[+0.190]	+32.184	......	+7.951	......	+35.511	+83.787
VIII	V''' + VI'' + VII' + 9	......	......	......	......	......	......	......	......	+30.488	+0.027	+7.969	......	+38.343	+76.828

TABLE 15.—(*Continued*).

Number of equation.	Remarks.	ϕ_1.	ϕ_2.	ϕ_3.	ϕ_4.	ϕ_5.	ϕ_6.	ϕ_7.	ϕ_9.	ϕ_8.	ϕ_{11}.	ϕ_{10}.	ϕ_{12}.	Absolute term.	Check.
VII''	VII $\times -\frac{2.501}{18.538}$, *i. e.*, 0.13491	......	......	......	......	......	......	......	[−2.501]	[+0.027]	−0.337	−0.227	......	−2.531	−5.570
VIII'	VIII $\times -\frac{0.027}{30.488}$, *i. e.*, 0.00089	......	......	......	......	......	......	......	......	[−0.027]	−0.000	−0.007	......	−0.034	−0.068
10	......	......	......	......	......	......	......	......	[+2.501]	......	+10.610	+0.303	+2.501	+27.273	+43.188
IX	VII'' + VIII' + 10	......	......	......	......	......	......	......	......	......	+10.273	+0.069	+2.501	+24.708	+37.550
VII'''	VII $\times -\frac{1.685}{18.538}$, *i. e.*, 0.09089	......	......	......	......	......	......	......	[−1.685]	[+0.018]	[−0.227]	−0.153	......	−1.765	−3.753
VIII''	VIII $\times -\frac{7.969}{30.488}$, *i. e.*, 0.26138	......	......	......	......	......	......	......	......	[−7.969]	[−0.007]	−2.083	......	−10.022	−20.081
IX'	IX $\times -\frac{0.069}{10.278}$, *i. e.*, 0.00672	......	......	......	......	......	......	......	......	......	[−0.069]	−0.000	−0.017	−0.166	−0.252
11	......	......	......	......	......	......	......	......	[+1.685]	[+7.951]	[+0.303]	+30.806	+5.464	+71.556	+117.765
X	VII''' + VIII'' + IX' + 11	......	......	......	......	......	......	......	......	......	......	+28.570	+5.447	+59.663	+93.679
IX''	IX $\times -\frac{2.501}{10.273}$, *i. e.*, 0.24345	......	......	......	......	......	......	......	......	......	[−2.501]	[−0.017]	−0.609	−6.015	−9.142
X'	X $\times -\frac{5.447}{28.570}$, *i. e.*, 0.19065	......	......	......	......	......	......	......	......	......	......	[−5.447]	−1.038	−11.375	−17.860
12	......	......	......	......	......	......	......	......	......	......	[+2.501]	[+5.464]	+15.930	+38.178	+62.073
XI	IX'' + X' + 12	......	......	......	......	......	......	......	......	......	......	......	+14.283	+20.788	+35.071
		+0.86345	+0.90366	+1.05699	+0.79069	+0.59943	+0.52612	+0.56932	−0.58077	−0.78252	−2.03864	−1.81082	−1.45544	......	......

TABLE 16.—MOHR'S SEMI-GRAPHICAL METHOD. SECONDARY STRESSES IN MEMBERS.

Joint. (1)	Member. (2)	$\frac{2c}{l}$. (3)	ϕ. (4)	-3ψ. (5)	$(2\phi_{an}+\phi_n-3\psi_{an})$. (6)	f (Column 3 × Column 6) in pounds per square inch. (7)
1	1- 2	0.03926 0.03972	+0.863	−2.526	+0.105	− 0.41 + 0.42
	1- 3	0.05		−3.012	−0.228	± 1.14
2	2- 4	0.0622 0.0636	+0.904	−2.820	−0.222	+ 1.38 − 1.41
	2- 5	0.03139		−2.010	+0.397	± 1.25
	2- 3	0.02839		−1.851	+1.013	∓ 2.88
	2- 1	0.03972 0.03926		−2.526	+0.145	− 0.58 + 0.57
3	3- 1	0.05	+1.0570	−3.012	−0.035	± 0.18
	3- 2	0.02839		−1.851	+1.167	∓ 3.31
	3- 5	0.05		−2.820	−0.107	± 0.54
4	4- 6	0.0622 0.0636	+0.791	−1.800	+0.308	− 1.92 + 1.96
	4- 5	0.02790		−1.032	+1.149	∓ 3.21
	4- 2	0.0636 0.0622		−2.820	−0.335	+ 2.13 − 2.08
5	5- 3	0.05	+0.599	−2.820	−0.564	± 2.82
	5- 2	0.03139		−2.010	+0.093	∓ 0.29
	5- 4	0.02790		−1.032	+0.958	∓ 2.67
	5- 6	0.03139		−1.065	+0.660	∓ 2.07
	5- 7	0.05		−1.800	−0.032	± 0.16
6	6- 8	0.0622 0.0636	+0.526	−0.636	−0.366	+ 2.28 − 2.33
	6- 9	0.03139		+0.099	+0.569	∓ 1.79
	6- 7	0.02839		......	+1.622	∓ 4.60
	6- 5	0.03139		−1.065	+0.587	∓ 1.84
	6- 4	0.0636 0.0622		−1.800	+0.043	− 0.27 + 0.27
7	7- 5	0.05	+0.569	−1.800	−0.062	± 0.31
	7- 6	0.02839		......	+1.665	∓ 4.73
	7- 9	0.05		−0.636	−0.079	± 0.39
8	8-10	0.0622 0.0636	−0.783	+5.067	+1.691	−10.52 +10.75
	8- 9	0.02790		+1.482	−0.665	± 1.86
	8- 6	0.0636 0.0622		−0.636	−1.675	+10.65 −10.42
9	9- 7	0.05	−0.582	−0.636	−1.230	± 6.15
	9- 6	0.03139		+0.099	−0.538	± 1.69
	9- 8	0.02790		+1.482	−0.464	± 1.29
	9-10	0.03139		+3.444	+0.470	∓ 1.47
	9-11	0.05		+5.067	+1.865	∓ 9.33
10	10-12	0.03926 0.03972	−1.811	+4.488	−0.589	+ 2.31 − 2.34
	10-11	0.02839		+3.120	−2.540	± 7.21
	10- 9	0.03139		+3.444	−0.759	± 2.38
	10- 8	0.0636 0.0622		+5.067	+0.663	− 4.22 + 4.12
11	11- 9	0.05	−2.039	+5.067	+0.408	∓ 2.04
	11-10	0.02839		+3.120	−2.768	± 7.86
	11-12	0.05		+5.460	−0.073	± 0.36
12	12-11	0.05	−1.455	+5.460	+0.510	∓ 2.55
	12-10	0.03972 0.03926		+4.488	−0.234	+ 0.93 − 0.92

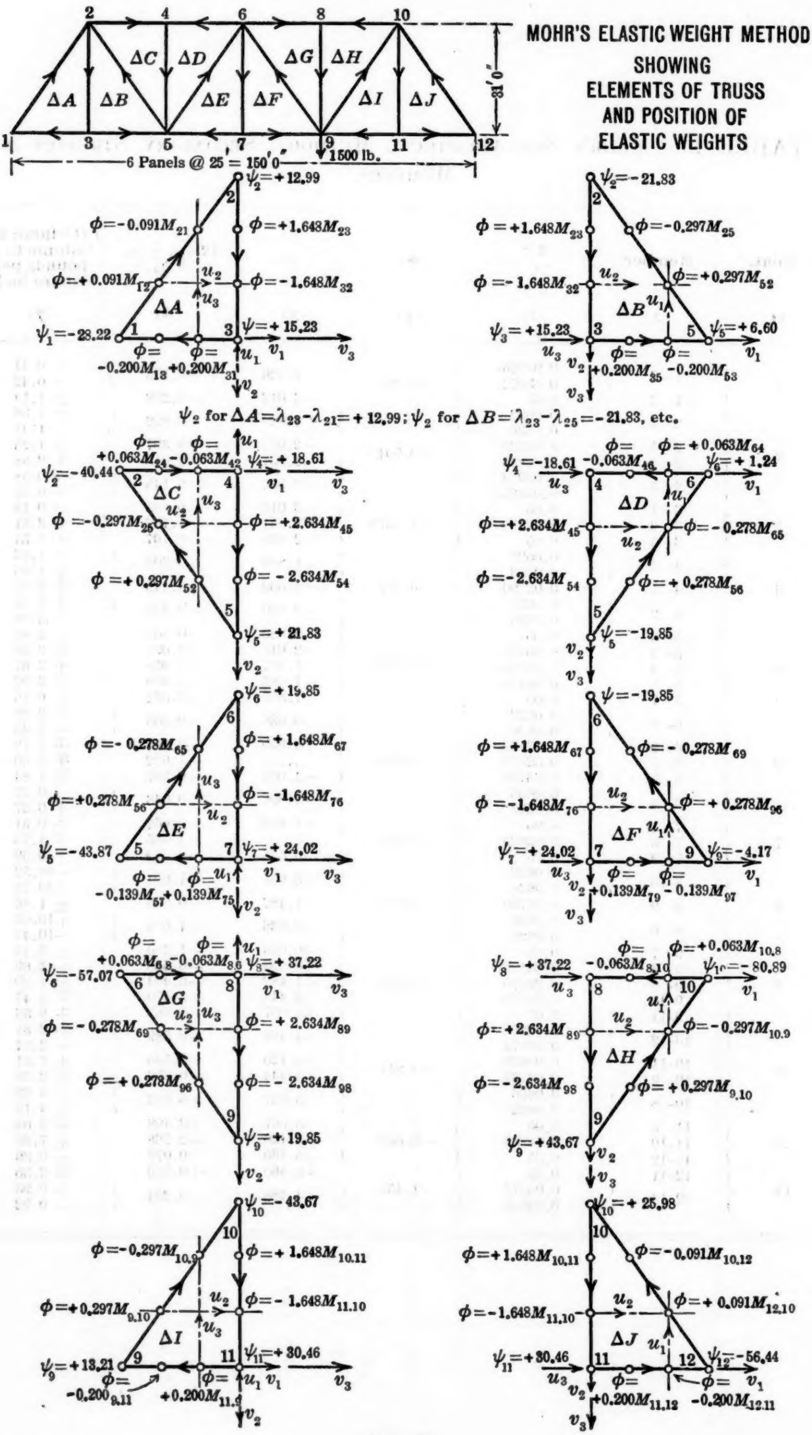

MOHR'S ELASTIC WEIGHT METHOD
SHOWING
ELEMENTS OF TRUSS
AND POSITION OF
ELASTIC WEIGHTS
31'0"
6 Panels @ 25' = 150'0"
1500 lb.
ΔA
ΔB
ΔC
ΔD
ΔE
ΔF
ΔG
ΔH
ΔI
ΔJ
ψ2 for ΔA=λ23-λ21=+12.99; ψ2 for ΔB=λ23-λ25=-21.83, etc.

FIG. 55.

TABLE 17.—MOHR'S ELASTIC WEIGHT METHOD. ELEMENTS OF MEMBERS.

Member.	*l*.	*A*.	*S*.	*I*.	λ.	β.
1-3	300	26.48	+ 403.25	750.2	+15.23	0.200
3-5	"	"	"	"	"	"
5-7	"	50.36	+1209.7	1076.4	+24.02	0.139
7-9	"	"	"	"	"	"
9-11	"	26.48	+ 806.45	750.2	+30.46	0.200
11-12	"	"	"	"	"	"
1-2	477.9	49.45	− 642.35	2611.53	−12.99	0.091
2-4	300	43.33	− 806.45	2385.38	−18.61	0.063
4-6	"	"	"	"	"	"
6-8	"	"	−1612.9	"	−37.22	"
8-10	"	"	"	"	"	"
10-12	477.9	49.45	−1284.65	2611.53	−25.98	0.091
2-3	372	24.35	....	112.86	....	1.648
4-5	"	12.20	....	70.62	....	2.634
6-7	"	24.35	....	112.86	....	1.648
8-9	"	12.20	....	70.62	....	2.634
10-11	"	24.35	....	112.86	....	1.648
2-5	477.9	29.42	+ 642.35	805.4	+21.83	0.297
5-6	"	32.36	− 642.35	860.4	−19.85	0.278
6-9	"	"	+ 642.35	"	+19.85	"
9-10	"	29.42	+1284.65	805.4	+43.67	0.297

E = Modulus of elasticity, taken as unity.
A = Area of cross-section of member, in square inches.
S = Total load in member, in pounds.
l = Length of member, in inches.
I = Moment of inertia of member, in inch units.
$\lambda = \frac{S}{EA}$, strain (*Dehnung*).
$\beta = \frac{l}{2\,EI}$, elastic weight (*Masse der Biegsamheit*).

of the triangles is 31 ft., while the base is 25 ft.; hence, it is that the factors, $\frac{25}{31}$ and $\frac{31}{25}$, appear in the ψ-terms of Table 18.

Table 19 is only a re-arrangement of the equations of Table 18. Table 20 is the result of expressing every M in terms of M_{12} and M_{21}, which is possible by starting at the beginning of Table 19.

It will be found that $\Sigma M_{10} = \Sigma M_{12} = 0$ are two equations that have not been used at all. After solving these, for they are simultaneous linear equations in M_{12} and M_{21}, Table 21 and, finally, Table 22 follow.

It must be noted that positive moments are counterclockwise, hence, the signs are as shown in front of the various secondary stresses.

The results in Table 19 and the following tables have been computed each time to five places of decimals. The writer believed that a good basis for accuracy in all the methods would be to carry out Mohr's elastic weight method to ten places of decimals, the computing machine making this quite possible. It goes without saying that no engineer demands such accuracy; the only justification in the present case is that the method which gave results closest to what may be termed the standard in this investigation was the best for purposes of accuracy combined with speed in obtaining the secondary stresses.

Mao's Graphical Method of Deformation Contour.—Figs. 56 and 57 and Table 23 represent Mao's graphical method applied to the problem. On the truss diagram, Fig. 56 (*a*), the moments of inertia, *I*, and the unit stresses, *P*,

TABLE 18.—MOHR'S ELASTIC WEIGHT METHOD. MOMENT EQUATIONS, ASSUMING M_{12} AND M_{21} TO BE KNOWN.

I 1	$0 = M_{12} + M_{31}$
II 1	$0 = -0.200\,M_{31} + 2 \times 0.200\,M_{8} - 2 \times 0.091\,M_{12} + 0.091\,M_{21} - 12.99 \times 3 \times \frac{31}{25}$
2	$0 = +0.091\,M_{21} - 1.648\,M_{23} + 0.200\,M_{31} - 0.200\,M_{13} - 28.22 \times 3 \times \frac{25}{31}$
3	$0 = +0.200\,M_{13} - 0.091\,M_{12} + 1.648\,M_{23} - 1.648\,M_{32} - 12.99 \times 3 \times \frac{31}{25}$
I 3	$0 = M_{31} + M_{32} + M_{35}$
II 4	$0 = -0.200\,M_{35} + 2 \times 1.648\,M_{32} - 2 \times 1.648\,M_{23} + 0.297\,M_{25} + 21.85 \times 3 \times \frac{31}{25}$
5	$0 = -1.648\,M_{23} + 0.297\,M_{25} - 0.200\,M_{53} + 0.200\,M_{35} - 6.60 \times 3 \times \frac{25}{31}$
6	$0 = +1.648\,M_{32} - 2 \times 1.648\,M_{23} + 2 \times 0.297\,M_{25} - 0.297\,M_{52} - 6.60 \times 3 \times \frac{25}{31}$
I 2	$0 = M_{24} + M_{25} + M_{23} + M_{21}$
II 7	$0 = -0.297\,M_{52} + 2 \times 0.297\,M_{25} - 2 \times 0.063\,M_{24} + 0.063\,M_{42} + 21.83 \times 3 \times \frac{31}{25}$
8	$0 = -0.063\,M_{24} + 0.063\,M_{42} - 2.634\,M_{54} + 0.297\,M_{52} + 40.44 \times 3 \times \frac{25}{31}$
9	$0 = +0.297\,M_{25} - 0.063\,M_{24} + 2.634\,M_{45} - 2.634\,M_{54} + 21.83 \times 3 \times \frac{31}{25}$
I 4	$0 = M_{46} + M_{45} + M_{42}$
II 10	$0 = -0.278\,M_{56} + 2 \times 2.634\,M_{54} - 2 \times 2.634\,M_{45} + 0.063\,M_{46} - 19.85 \times 3 \times \frac{31}{25}$
11	$C = +0.063\,M_{46} - 0.063\,M_{64} + 0.278\,M_{56} - 2.634\,M_{54} - 1.24 \times 3 \times \frac{25}{31}$
12	$0 = -0.278\,M_{65} + 2 \times 0.278\,M_{56} - 2 \times 2.634\,M_{54} + 2.634\,M_{45} - 1.24 \times 3 \times \frac{25}{31}$
I 5	$0 = M_{53} + M_{52} + M_{54} + M_{56} + M_{57}$
II 13	$0 = -0.139\,M_{75} + 2 \times 0.139\,M_{57} - 2 \times 0.278\,M_{56} + 0.278\,M_{65} - 19.85 \times 3 \times \frac{31}{25}$
14	$0 = +0.278\,M_{65} - 1.648\,M_{67} + 0.139\,M_{75} - 0.139\,M_{57} - 43.87 \times 3 \times \frac{25}{31}$
15	$0 = +0.139\,M_{57} - 0.278\,M_{56} + 1.648\,M_{67} - 1.648\,M_{76} - 19.85 \times 3 \times \frac{31}{25}$
I 7	$0 = M_{75} + M_{76} + M_{79}$
II 16	$0 = -0.139\,M_{79} + 2 \times 1.648\,M_{76} - 2 \times 1.648\,M_{67} + 0.278\,M_{69} + 19.85 \times 3 \times \frac{31}{25}$
17	$0 = -1.648\,M_{67} + 0.278\,M_{69} - 0.139\,M_{97} + 0.139\,M_{79} + 4.17 \times 3 \times \frac{25}{31}$
18	$0 = +1.648\,M_{76} - 2 \times 1.648\,M_{67} + 2 \times 0.278\,M_{69} - 0.278\,M_{96} + 4.17 \times 3 \times \frac{25}{31}$
I 6	$0 = M_{68} + M_{69} + M_{67} + M_{65} + M_{64}$
II 19	$0 = -0.278\,M_{96} + 2 \times 0.278\,M_{69} - 2 \times 0.063\,M_{68} + 0.063\,M_{86} + 19.85 \times 3 \times \frac{31}{25}$
20	$0 = -0.063\,M_{68} + 0.063\,M_{86} - 2.634\,M_{98} + 0.278\,M_{96} - 57.07 \times 3 \times \frac{25}{31}$
21	$0 = +0.278\,M_{69} - 0.063\,M_{68} + 2.634\,M_{89} - 2.634\,M_{98} + 19.85 \times 3 \times \frac{25}{31}$
I 8	$0 = M_{8.10} + M_{89} + M_{86}$
II 22	$0 = -0.297\,M_{9.10} + 2 \times 2.634\,M_{98} - 2 \times 2.634\,M_{89} + 0.063\,M_{8.10} + 43.67 \times 3 \times \frac{31}{25}$
23	$0 = +0.063\,M_{8.10} - 0.063\,M_{10.8} + 0.297\,M_{9.10} - 2.634\,M_{98} + 80.89 \times 3 \times \frac{25}{31}$
24	$0 = -0.297\,M_{10.9} + 2 \times 0.297\,M_{9.10} - 2 \times 2.634\,M_{98} + 2.634\,M_{89} + 80.89 \times 3 \times \frac{25}{31}$

TABLE 18.—(*Continued.*)

I 9	$0 = M_{97} + M_{96} + M_{98} + M_{9.10} + {}_{9.11}$
II 25	$0 = -0.200\ M_{11.9} + 2 \times 0.200\ M_{9.11} - 2 \times 0.297\ M_{9.10} + 0.297\ M_{10.9} + 43.67 \times 3 \times \frac{31}{25}$
26	$0 = +0.297\ M_{10.9} - 1.648\ M_{10.11} + 0.200\ M_{11.9} - 0.200\ M_{9.11} + 13.21 \times 3 \times \frac{25}{31}$
27	$0 = +0.200\ M_{9.11} - 0.297\ M_{9.10} + 1.648\ M_{10.11} - 1.648\ M_{11.10} + 43.67 \times 3 \times \frac{31}{25}$
I 10	$0 = M_{10.12}\ M_{10.11} + M_{10.9} + M_{10.8}$
II 28	$0 = -0.200\ M_{11.12} + 3 \times 1.648\ M_{11.10} - 2 \times 1.648\ M_{10.11} + 0.091\ M_{10.12} - 25.98 \times 3 \times \frac{31}{25}$
29	$0 = -1.648\ M_{10.11} + 0.091\ M_{10.12} - 0.200\ M_{12.11} + 0.200\ M_{11.12} + 56.44 \times 3 \times \frac{25}{31}$
30	$0 = +1.648\ M_{11.10} - 2 \times 1.648\ M_{10.11} + 2 \times 0.091\ M_{10.12} - 0.091\ M_{12.10} + 56.44 \times 3 \times \frac{25}{31}$
I 11	$0 = M_{11.9} + M_{11.10} + M_{11.12}$
12	$0 = M_{12.11} + M_{12.10}$

TABLE 19.—MOHR'S ELASTIC WEIGHT METHOD. MOMENT EQUATIONS.

I 1	$M_{13} = -M_{12}$
II 1	$M_{31} = +2M_{13} - 0.91M_{12} + 0.455M_{21} - 241.614$
2	$M_{23} = +0.05522M_{21} + 0.12136M_{31} - 0.12136M_{13} - 41.42852$
3	$M_{32} = +0.12136M_{13} - 0.05522M_{12} + M_{23} - 29.32208$
I 3	$M_{35} = -M_{31} - M_{32}$
II 4	$M_{25} = +0.67340M_{35} - 11.09764M_{32} + 11.09764M_{23} - 273.42626$
5	$M_{53} = -8.24M_{23} + 1.485M_{25} + M_{35} - 79.83871$
6	$M_{52} = +5.54882M_{32} - 11.09764M_{23} + 2M_{25} - 53.76344$
I 2	$M_{24} = -M_{25} - M_{23} - M_{21}$
II 7	$M_{42} = +4.71429M_{52} - 9.42857M_{25} + 2M_{24} - 1289.00952$
8	$M_{54} = -0.02392M_{24} + 0.02392M_{42} + 0.11276M_{52} - 37.14454$
9	$M_{45} = -0.11276M_{25} + 0.02392M_{24} + M_{54} - 30.83053$
I 4	$M_{46} = -M_{45} - M_{42}$
II 10	$M_{56} = +18.94964M_{54} - 18.94964M_{45} + 0.22662M_{46} - 265.61870$
11	$M_{64} = +M_{46} + 4.41270M_{56} - 41.80952M_{54} - 47.61905$
12	$M_{65} = +2M_{56} - 18.94964M_{54} + 9.47482M_{45} - 10.79136$
I 5	$M_{57} = -M_{53} - M_{52} - M_{54} - M_{56}$
II 13	$M_{75} = +2M_{57} - 4M_{56} + 2M_{65} - 531.23741$
14	$M_{67} = +0.16869M_{65} + 0.08434M_{75} - 0.08434M_{57} - 64.40358$
15	$M_{76} = +0.08434M_{57} - 0.16869M_{56} + M_{67} - 44.80704$
I 7	$M_{79} = -M_{75} - M_{76}$
II 16	$M_{69} = +0.5M_{79} - 11.85612M_{76} + 11.85612M_{67} - 265.61870$
17	$M_{97} = -11.85612M_{67} + 2M_{69} + M_{79} + 72.58065$
18	$M_{96} = +5.92806M_{76} - 11.85612M_{67} + 2M_{69} + 36.29032$
I 6	$M_{68} = -M_{69} - M_{67} - M_{65} - M_{64}$
II 19	$M_{86} = +4.41270M_{96} - 8.82540M_{69} + 2M_{68} - 1172.09524$
20	$M_{98} = -0.02392M_{68} + 0.02392M_{86} + 0.10554M_{96} - 52.41985$
21	$M_{89} = -0.10554M_{69} + 0.02392M_{68} + M_{98} - 28.03417$
I 8	$M_{8.10} = -M_{89} - M_{86}$
II 22	$M_{9.10} = +17.73737M_{98} - 17.73737M_{89} + 0.21212M_{8.10} + 546.97778$
23	$M_{10.8} = +M_{8.10} + 4.71429M_{9.10} - 41.80952M_{98} + 3106.37481$
24	$M_{10.9} = +2M_{9.10} - 17.73737M_{98} + 8.86869M_{89} + 658.92799$
I 9	$M_{9.11} = -M_{97} - M_{96} - M_{98} - M_{9.10}$
II 25	$M_{11.9} = +2M_{9.11} - 2.97M_{9.10} + 1.485M_{10.9} + 812.262$
26	$M_{10.11} = +0.18022M_{10.9} + 0.12136M_{11.9} - 0.12136M_{9.11} + 19.39301$
27	$M_{11.10} = +0.12136M_{9.11} - 0.18022M_{9.10} + M_{10.11} + 98.57549$
I 10	$M_{10.12} = -M_{10.11} - M_{10.9} - M_{10.8}$
II 28	$M_{11.12} = +16.48M_{11.10} - 16.48M_{10.11} + 0.455M_{10.13} - 483.228$
29	$M_{12.11} = -8.24M_{10.11} + 0.455M_{10.12} + M_{11.12} + 682.74194$
30	$M_{12.10} = +18.10989M_{11.10} - 36.21978M_{10.11} + 2M_{10.12} + 1500.53173$

are given, also, the U-lines, which are drawn from one end of a member tangent to the circle the center of which is at the other end, and the radius of which is equal to the extreme fiber distance for the member. For example, there is such a line from Joint (1) to the circle of radius, $y = 9.38$ in. at Joint (2). Finally, by Ritter's method the half values of the changes of angle are graphically determined on this diagram. It has been shown (see page 1010) that the scales call for the half values of the changes of angle when the lengths of the members are expressed in feet and the fiber distances in inches.

TABLE 20.—MOHR'S ELASTIC WEIGHT METHOD. MOMENTS IN TERMS OF M_{12} AND M_{21}.

I	1	M_{13}	$= -\ M_{12}$		
II	1	M_{31}	$= -\ 2.91M_{12}$	$+\ 0.455M_{21}$	$-\ 241.614$
	2	M_{23}	$= -\ 0.23180M_{12}$	$+\ 0.11044M_{21}$	$-\ 70.75080$
	3	M_{32}	$= -\ 0.40838M_{12}$	$+\ 0.11044M_{21}$	$-\ 100.07288$
II	3	M_{35}	$= +\ 3.31838M_{12}$	$-\ 0.56544M_{21}$	$+\ 341.68688$
I	4	M_{25}	$= +\ 4.19422M_{12}$	$-\ 0.38077M_{21}$	$+\ 282.05736$
	5	M_{53}	$= +\ 11.45683M_{12}$	$-\ 2.04091M_{21}$	$+\ 1\,263.68994$
	6	M_{52}	$= +\ 8.69485M_{12}$	$-\ 1.37435M_{21}$	$+\ 740.23179$
I	2	M_{24}	$= -\ 3.96242M_{12}$	$-\ 0.72967M_{21}$	$-\ 211.30656$
II	7	M_{42}	$= -\ 6.48030M_{12}$	$-\ 4.34830M_{21}$	$-\ 881.35287$
	8	M_{54}	$= +\ 0.92020M_{12}$	$-\ 0.24153M_{21}$	$+\ 30.29649$
	9	M_{45}	$= +\ 0.35248M_{12}$	$-\ 0.21604M_{21}$	$-\ 37.39328$
I	4	M_{46}	$= +\ 6.12782M_{12}$	$+\ 4.56434M_{21}$	$+\ 918.74615$
II	10	M_{56}	$= +\ 12.14678M_{12}$	$+\ 0.55134M_{21}$	$+\ 1\,225.28432$
	11	M_{64}	$= +\ 21.25480M_{12}$	$+\ 17.09549M_{21}$	$+\ 5\,011.25752$
	12	M_{65}	$= +\ 10.19578M_{12}$	$+\ 3.63265M_{21}$	$+\ 1\,511.37510$
I	5	M_{57}	$= -\ 33.21866M_{12}$	$+\ 3.10545M_{21}$	$-\ 3\,259.50254$
II	13	M_{75}	$= -\ 94.63288M_{12}$	$+\ 11.27084M_{21}$	$-\ 8\,928.62957$
	14	M_{67}	$= -\ 3.45975M_{12}$	$+\ 1.30146M_{21}$	$-\ 287.58388$
	15	M_{76}	$= -\ 8.31045M_{12}$	$+\ 1.47036M_{21}$	$-\ 813.99057$
I	7	M_{79}	$= +\ 102.94333M^{12}$	$-\ 12.74120M_{21}$	$+\ 9\,742.62014$
II	16	M_{69}	$= +\ 108.98214M_{12}$	$-\ 8.37310M_{21}$	$+\ 10\,846.83226$
	17	M_{97}	$= +\ 361.92682M_{12}$	$-\ 44.91767M_{21}$	$+\ 34\,918.49430$
	18	M_{96}	$= +\ 209.71864M_{12}$	$-\ 23.46009M_{21}$	$+\ 20\,314.19889$
I	6	M_{68}	$= -\ 136.97297M_{12}$	$-\ 13.65650M_{21}$	$-\ 17\,081.88100$
II	19	M_{86}	$= -\ 310.33148M_{12}$	$-\ 56.93938M_{21}$	$-\ 41\,423.02523$
	20	M_{98}	$= +\ 17.98697M_{12}$	$-\ 3.51131M_{21}$	$+\ 1\,509.30103$
	21	M_{89}	$= +\ 3.20860M_{12}$	$-\ 2.95427M_{21}$	$-\ 72.10640$
I	8	$M_{8.10}$	$= +\ 307.12288M_{12}$	$+\ 59.89365M_{21}$	$+\ 41\,495.13163$
II	22	$M_{9.10}$	$= +\ 327.27633M_{12}$	$+\ 2.82422M_{21}$	$+\ 37\,398.93381$
	23	$M_{10.8}$	$= +1\,097.97183M_{12}$	$+\ 220.01403M_{21}$	$+157\,807.77451$
	24	$M_{10.9}$	$= +\ 363.96720M_{12}$	$+\ 41.72934M_{21}$	$+\ 48\,046.27549$
I	9	$M_{9.11}$	$= -\ 916.90876M_{12}$	$+\ 69.06485M_{21}$	$-\ 94\,140.92803$
II	25	$M_{11.9}$	$= -2\,265.77909M_{12}$	$+\ 191.70984M_{21}$	$-227\,195.70838$
	26	$M_{10.11}$	$= -\ 98.10473M_{12}$	$+\ 22.40466M_{21}$	$-\ 7\,469.23536$
	27	$M_{11.10}$	$= -\ 268.36252M_{12}$	$+\ 30.27739M_{21}$	$-\ 25\,535.63875$
I	10	$M_{10.12}$	$= -1\,363.83430M_{12}$	$-\ 284.14803M_{21}$	$-198\,384.81464$
II	28	$M_{11.12}$	$= -3\,426.39299M_{12}$	$+\ 0.45524M_{21}$	$-388\,482.64653$
	29	$M_{12.11}$	$= -3\,238.55460M_{12}$	$-\ 313.44651M_{21}$	$-416\,518.49588$
	30	$M_{12.10}$	$= -4\,034.35258M_{12}$	$-\ 831.46771M_{21}$	$-587\,182.64488$

Equations I (11) and I (12) give:

$$0 = -5\,960.53460M_{12} + 222.44247M_{21} - 641\,213.99366$$
$$0 = -7\,272.90718M_{12} - 1\,144.91422M_{21} - 1\,003\,701.14076,$$

that is:

$$0 = -26.795848M_{12} + M_{21} - 2\,882.605977$$
$$0 = -6.352360M_{12} - M_{21} - 876.660560,$$

that is:

$$M_{12} = -113.407836,$$
$$M_{21} = -156.253159.$$

In Fig. 56(b) is shown the force diagram of the section moduli, these being the forces the distances of which from the joints will be the secondary stresses. These S-forces are drawn in directions perpendicular to their various members. The broken lines represent the directions that the R-forces take at the various joints, in order to preserve equilibrium with the S-forces.

The property and deformation lines are then constructed (Fig. 57). All the sets of property lines but one are omitted because any more lines than those given would make the diagram rather confusing. The lines marked K_{A1}, K_{A2}, etc., are the deformation lines in position (see page 1013). Then, all the base and vertex lines of all the triangles are located, the full lines being the base lines and the broken lines the vertex lines (see page 1019).

TABLE 21.—MOHR'S ELASTIC WEIGHT METHOD. VALUES OF MOMENTS.

I	1	M_{13}	= + 113.408			= + 113.408
II	1	M_{31}	= + 330.01680	− 71.09519	− 241.614	= + 17.808
	2	M_{23}	= + 26.28794	− 17.25660	− 70.75080	= − 61.719
	3	M_{32}	= + 46.31349	− 17.25660	− 100.07288	= − 71.016
I	3	M_{35}	= − 376.33029	+ 88.35179	+ 341.68686	= + 53.708
II	4	M_{25}	= − 475.65741	+ 59.49652	+ 282.05736	= − 134.104
	5	M_{53}	= − 1 299.29430	+ 318.89862	+ 1 263.68994	= + 283.294
	6	M_{52}	= − 986.06412	+ 214.74653	+ 740.23179	= − 31.086
I	2	M_{24}	= + 449.36948	+ 114.01324	− 211.30656	= + 352.076
II	7	M_{42}	= + 734.91680	+ 679.43561	− 881.35287	= + 533.000
	8	M_{54}	= − 104.35789	+ 37.73983	+ 30.29649	= − 36.322
	9	M_{45}	= − 39.97399	+ 33.75693	− 37.39328	= − 43.610
I	4	M_{46}	= − 694.94281	− 713.19254	+ 918.74615	= − 489.389
II	10	M_{56}	= − 1 377.54003	− 86.14862	+ 1 225.28432	= − 238.404
	11	M_{64}	= − 2 410.46087	− 2 671.22432	+ 5 011.25752	= − 70.428
	12	M_{65}	= − 1 156.28135	− 567.61304	+ 1 511.37510	= − 212.519
I	5	M_{57}	= + 3 767.25635	− 485.23637	− 3 259.50254	= + 22.517
II	13	M_{75}	= + 10 732.11014	− 1 761.10485	− 8 928.62957	= + 42.376
	14	M_{67}	= + 392.36276	− 203.35724	− 287.58388	= − 98.578
	15	M_{76}	= + 942.46815	− 229.74839	− 813.99057	= − 101.271
I	7	M_{79}	= − 11 674.58029	+ 1 990.85275	+ 9 742.62014	= + 58.893
II	16	M_{69}	= − 12 359.42866	+ 1 308.32333	+ 10 846.83226	= − 204.273
	17	M_{97}	= − 41 045.31530	+ 7 018.52783	+ 34 918.49430	= + 891.707
	18	M_{96}	= − 23 783.73713	+ 3 665.71817	+ 20 314.19889	= + 196.175
I	6	M_{68}	= + 15 533.80812	+ 2 133.87127	− 17 081.88100	= + 585.798
II	19	M_{86}	= + 35 194.02159	+ 8 896.95800	− 41 423.02523	= + 2 667.954
	20	M_{98}	= − 2 039.86334	+ 548.65328	+ 1 509.30103	= + 18.091
	21	M_{89}	= − 363.88038	+ 461.61402	− 72.10640	= + 25.627
I	8	$M_{8.10}$	= − 34 830.14121	− 9 358.57202	+ 41 495.13163	= − 2 693.582
II	22	$M_{9.10}$	= − 37 115.70036	− 441.29830	+ 37 398.93381	= − 158.060
	23	$M_{10.8}$	= − 124 518.60923	− 34 377.88721	+ 157 807.77451	= − 1 088.722
	24	$M_{10.9}$	= − 41 276.73253	− 6 520.34120	+ 48 046.27549	= + 249.202
I	9	$M_{9.11}$	= + 103 984.63828	− 10 791.60099	− 94 140.92803	= − 947.892
II	25	$M_{11.9}$	= + 256 957.10345	− 29 955.26811	− 227 195.70838	= − 193.873
	26	$M_{10.11}$	= + 11 125.84513	− 3 500.79890	− 7 469.23536	= + 155.811
	27	$M_{11.10}$	= + 30 434.41266	− 4 730.93788	− 25 535.63875	= + 167.836
I	10	$M_{10.12}$	= + 154 669.49663	+ 44 399.02731	− 198 384.81464	= + 683.709
II	28	$M_{11.12}$	= + 388 579.81428	− 71.13269	− 388 482.64653	= + 26.035
	29	$M_{12.11}$	= + 367 277.46895	+ 48 977.00737	− 416 518.49588	= − 264.020
	30	$M_{12.10}$	= + 457 527.19576	+ 129 919.45629	− 587 182.64488	= + 264.007

Following the instructions to determine the reference contour, $r_{11 \cdot 9}$ by means of the base and vertex lines of ΔJ, Vertex (11), *via* Vertex (12), and the base and vertex lines of ΔJ, Vertex (11), *via* Vertex (10), it was found that the fourth attempt gave an $r_{11 \cdot 9}$, that coincided with the value of $r_{11 \cdot 9}$, assumed before that trial was made. The shaded area around Joint (11) is the resulting deformation contour, while those around Joint (10) and Joint (12) had already been drawn in the final trial of determining $r_{11 \cdot 9}$.

A vertical from the end (*a*) of $r_{10 \cdot 8}$ to the base of ΔI, at (*b*), determines the point on the vertex line of ΔI, through which a parallel, (*b-d*), to the base line is drawn. A 45° line to this newly drawn line from the end (*c*) of $r_{11 \cdot 9}$ determines the point (*d*) from which the second 45° line is drawn to determine $r_{9 \cdot 11}$ (from 9 to *e*), which is negative in this case. The deformation contour is then drawn; it is shown by shading the edges. All the deformation contours are determined in a similar way. Finally, from the ends of r_{12} and

TABLE 22.—MOHR'S ELASTIC WEIGHT METHOD. SECONDARY STRESSES IN MEMBERS.

Joint.	Member.	Extreme fiber distance, C.	I, in inches⁴.	$\frac{C}{I}$.	M.	$f = \frac{C}{I} \times M$, in pounds per square inch.
1	1-2	9.38 9.49	2 611.5	0.003592 0.003634	− 113.408	− 0.41 + 0.41
	1-3	7.5	750.2	0.009997	+ 113.408	± 1.13
2	2-4	9.33 9.54	2 385.4	0.003911 0.003999	+ 352.076	+ 1.38 − 1.41
	2-5	7.5	805.4	0.009312	− 134.104	∓ 1.25
	2-3	5.28	112.86	0.046784	− 61.719	∓ 2.89
	2-1	9.49 9.38	2 611.5	0.003634 0.003592	− 156.253	− 0.57 + 0.56
3	3-1	7.5	750.2	0.009997	+ 17.308	± 0.17
	3-2	5.28	112.86	0.046784	− 71.016	∓ 3.32
	3-5	7.5	750.2	0.009997	+ 53.708	± 0.54
4	4-6	9.33 9.54	2 385.4	0.003911 0.003999	− 489.389	− 1.91 + 1.96
	4-5	5.19	70.62	0.073492	− 43.610	∓ 3.20
	4-2	9.54 9.33	2 385.4	0.003999 0.003911	+ 533.000	+ 2.13 − 2.08
5	5-3	7.5	750.2	0.009997	+ 283.294	± 2.83
	5-2	7.5	805.4	0.009312	− 31.086	∓ 0.29
	5-4	5.19	70.62	0.073492	− 36.322	∓ 2.67
	5-6	7.5	860.4	0.008717	− 238.404	∓ 2.08
	5-7	7.5	1 076.4	0.006968	+ 22.517	± 0.16
6	6-8	9.33 9.54	2 385.4	0.003911 0.003999	+ 585.798	+ 2.29 − 2.34
	6-9	7.5	860.4	0.008717	− 204.278	∓ 1.78
	6-7	5.28	112.86	0.046784	− 98.578	∓ 4.61
	6-5	7.5	860.4	0.008717	− 212.519	∓ 1.85
	6-4	9.54 9.33	2 385.4	0.003999 0.003911	− 70.428	− 0.28 + 0.28
7	7-5	7.5	1 076.4	0.006968	+ 42.576	± 0.30
	7-6	5.28	112.86	0.046784	− 101.271	∓ 4.74
	7-9	7.5	1 076.4	0.006968	+ 58.893	± 0.41
8	8-10	9.33 9.54	2 385.4	0.003911 0.003999	−2693.582	−10.53 +10.77
	8-9	5.19	70.62	0.073492	+ 25.627	± 1.88
	8-6	9.54 9.33	2 385.4	0.003999 0.003911	+2667.954	+10.67 −10.43
9	9-7	7.5	1 076.4	0.006968	+ 891.707	± 6.21
	9-6	7.5	860.4	0.008717	+ 196.175	± 1.71
	9-8	5.19	70.62	0.073492	+ 18.091	± 1.33
	9-10	7.5	805.4	0.009312	− 158.061	∓ 1.47
	9-11	7.5	750.2	0.009997	− 947.892	∓ 9.48
10	10-12	9.38 9.49	2 611.5	0.003592 0.003634	+ 683.709	+ 2.46 − 2.48
	10-11	5.28	112.86	0.046784	+ 155.811	± 7.29
	10-9	7.5	805.4	0.009312	+ 249.202	± 2.32
	10-8	9.54 9.33	2 385.4	0.003999 0.003911	−1088.722	− 4.35 + 4.26
11	11-9	7.5	750.2	0.009997	− 193.873	∓ 1.94
	11-10	5.28	112.86	0.046784	+ 167.836	± 7.85
	11-12	7.5	750.2	0.009997	+ 26.035	± 0.26
12	12-11	7.5	750.2	0.009997	− 264.020	∓ 2.64
	12-10	9.49 9.38	2 611.5	0.003634 0.003592	+ 264.007	+ 0.96 − 0.95

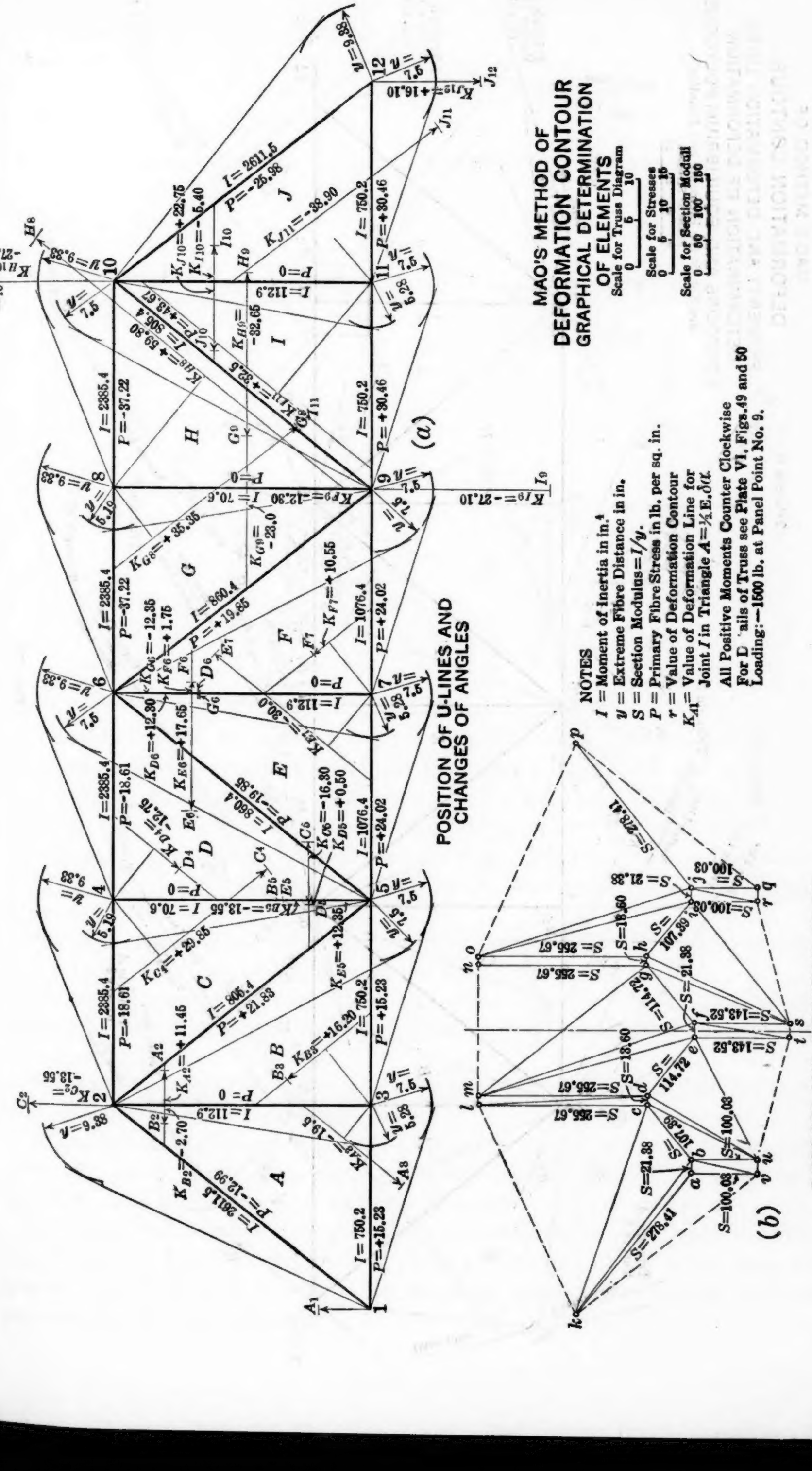

MAO'S METHOD OF
DEFORMATION CONTOUR
GRAPHICAL DETERMINATION
OF ELEMENTS
Scale for Truss Diagram 0 5 10
Scale for Stresses 0 5 10 15
Scale for Section Moduli 0 50 100 150
(a)
POSITION OF U-LINES AND
CHANGES OF ANGLES
NOTES
I = Moment of Inertia in in.^{4}
y = Extreme Fibre Distance in in.
S = Section Modulus = I/y.
P = Primary Fibre Stress in lb. per sq. in.
r = Value of Deformation Contour
K_{A1} = Value of Deformation Line for Joint 1 in Triangle A = ½E.δα.
All Positive Moments Counter Clockwise
For Details of Truss see Plate VI, Figs. 49 and 50
Loading: —1500 lb. at Panel Point No. 9.
(b)
S-FORCES AND DIRECTION OF R'S

FIG. 56.

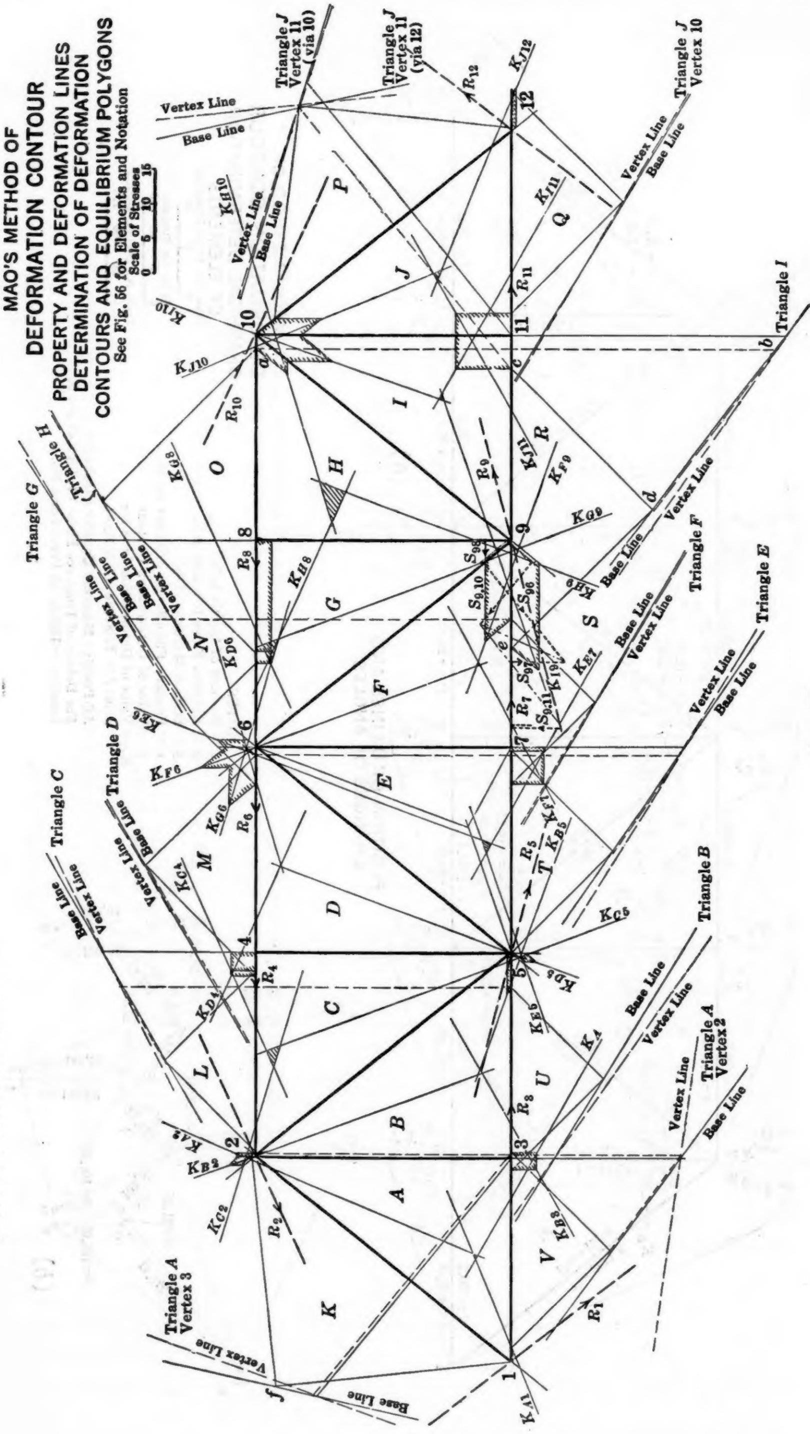

MAO'S METHOD OF
DEFORMATION CONTOUR
PROPERTY AND DEFORMATION LINES
DETERMINATION OF DEFORMATION
CONTOURS AND EQUILIBRIUM POLYGONS
See Fig. 56 for Elements and Notation
Scale of Stresses
0 5 10 15
Triangle A
Vertex 3
Vertex Line
Base Line
Triangle A
Vertex 2
Triangle B
Triangle C
Triangle D
Triangle E
Triangle F
Triangle G
Triangle H
Triangle I
Triangle J
Vertex 11
(via 10)
Triangle J
Vertex 11
(via 12)
Triangle J
Vertex 10

FIG. 57.

r_{21}, 45° lines to Member 1-2 are drawn, and it is found that their intersection, (*f*), practically falls on the parallel line to the base line of ΔA, Vertex (3), determined by drawing a perpendicular to Member 1-2 from the end of r_{31} and thus locating the point of intersection on the vertex line. This was a check on the work.

TABLE 23.—MAO'S METHOD OF DEFORMATION CONTOUR. SECONDARY STRESSES IN MEMBERS.

Joint.	Member.	*r's*, in pounds per square inch, from Fig. 51.	Secondary stress, in pounds per square inch.	Joint.	Member.	*r's*, in pounds per square inch, from Fig. 51.	Secondary stress, in pounds per square inch.
1	1- 2	− 0.25	− 0.42		7- 5	+ 0.55	+ 0.42
	1- 3	+ 2.15	+ 0.15	7	7- 6	− 4.75	− 4.67
	2- 4	+ 0.70	+ 1.40		7- 9	− 5.60	+ 0.25
2	2- 5	− 2.20	− 1.20		8- 6	+18.65	+10.53
	2- 3	− 2.50	− 2.98	8	8- 9	+ 2.45	+ 1.92
	2- 1	− 0.75	− 0.58		8-10	−17.00	−10.62
	3- 1	− 0.85	+ 0.15		9- 7	+11.95	+ 6.10
3	3- 2	− 3.80	− 3.37		9- 6	+ 5.30	+ 1.80
	3- 5	− 2.30	+ 0.23	9	9- 8	+ 0.85	+ 1.88
	4- 6	− 3.60	− 1.98		9-10	− 5.30	− 1.45
4	4- 5	− 3.75	− 3.23		9-11	−16.45	− 9.27
	4- 2	+ 2.80	+ 2.10		10-12	+ 3.60	+ 2.18
	5- 3	+ 5.30	+ 2.77		10-11	+ 6.95	+ 7.47
	5- 2	+ 0.80	− 0.20	10	10- 9	+ 6.25	+ 2.40
5	5- 4	− 2.20	− 2.72		10- 8	+ 2.15	− 4.23
	5- 6	− 2.30	− 2.10		11- 9	+ 5.10	− 2.08
	5- 7	+ 0.15	+ 0.28	11	11-10	+ 8.50	+ 7.98
	6- 8	− 5.70	+ 2.41		11-12	+ 3.35	+ 0.35
	6- 9	− 5.20	− 1.70	12	12-11	− 5.65	− 2.65
6	6- 7	− 4.50	− 4.58		12-10	− 0.65	+ 0.77
	6- 5	− 1.70	− 1.90				
	6- 4	+ 1.25	− 0.37				

As a final check, equilibrium polygons must be drawn around every joint. Only one, namely, at Joint (9), has been constructed. All the points of application of the *S*'s, except $S_{9\cdot11}$, were found from the deformation contour of Joint (9) and those of the surrounding joints. These *S*'s together with R_9 gave an $S_{9\cdot11}$ that lies only a short distance to the left of the $S_{9\cdot11}$, determined by $r_{9\cdot11}$ and $r_{11\cdot9}$. For the sake of clarity in the drawing the other equilibrium polygons are not shown.

There is a departure from Mao's method in that the *f*'s are determined from a table of *r*'s; Mao completes all the equilibrium polygons.

Mao's Analytical Method.—Plate VII gives the complete solution of the problem by means of Mao's analytical method with the rational arrangement of computations.

The explanations given previously (see page 1026 and Fig. 48) should serve to clarify the method of procedure here.

There is one departure from the original method; in the ΣM-table for Joint (6), there is an extra row, termed Σ, inserted between Row 6-7 and Row 6-9. The reason for this is to avoid errors. All the coefficients of the x and y and the constant terms for Rows 6-4, 6-5, and 6-7, are added. When Row 6-9 is completed, the negative of the sum of the x's of the Σ and Row 6-9 give the x-coefficient of Row 6-8. In the same manner, the y-coefficient and the k of Row 6-8 are obtained.

In the ΣM_{10} and ΣM_{12}-tables, the Σ is in the last row. An addition of the various columns gives the coefficients for the two simultaneous equations for finding x and y, that is, f_{12} and f_{21}, respectively. The little f-tables around the joints are then filled in, giving the secondary stresses.

Numerical Values.—For purposes of comparison, the amounts of secondary stresses by the foregoing methods have been collected in Table 24.

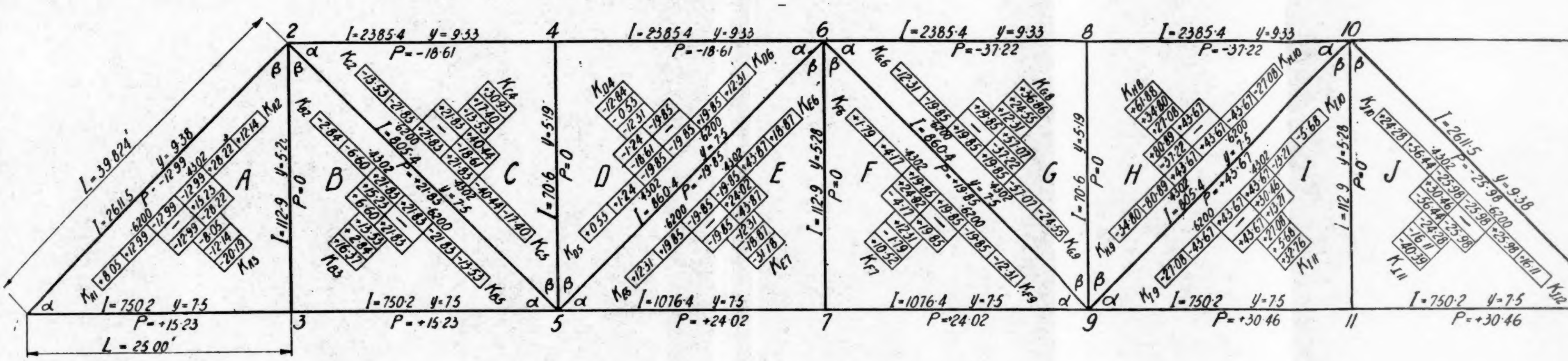
In determining the nature of the stress of an extreme fiber a clockwise rotation around each joint is made; a bending moment reckoned positive if counterclockwise then produces a tensile stress on the set of fibers first met and a compressive stress on those crossed last. Thus, at Joint (2), Member 2-4 is crossed over from top to bottom. It has a positive bending moment. The top fiber, therefore, has a secondary stress of — 1.35 lb. per sq. in. and the bottom one — 1.38 lb. per sq. in. (The difference in the numerical values is due to the respective fibers being 9.33 and 9.54 from the center of gravity of the member.) The resultant stresses of the top and bottom sets of fibers, therefore, would be (— 18.61 — 1.35) lb. per sq. in. and (— 18.61 — 1.38) lb. per sq. in., respectively.

For Member 2-3 at Joint (2) the set of fibers on the right-hand side (side first met) has a secondary stress of — 2.89 lb. per sq. in. and the left-hand side set has + 2.89 lb. per sq. in. The primary stress is zero and, therefore, the resultant stresses of the right and left-hand sets are — 2.89 and + 2.89 lb. per sq. in. The primary stresses are also given in Table 24 for information.

3.—Criticism of Methods

This criticism is a process of elimination, starting with Dr. Mao's methods which are believed to be very unsatisfactory.

Mao's Analytical Method.—Dr. Mao worked out the case for a symmetrical load. He was thereby able to solve the problem by using only half the number of triangles in the truss. The solution of the unsymmetrical case (Plate VII) makes possible conclusive statements with regard to certain features of this method. It shows how rapidly the coefficients of the x and y and the constant terms, k, increase, as triangle is added to triangle. The coefficients in the U-group on Member 1-2 are of the magnitude 0, 1, or 2, while those in the ΣM_{12}-group range from 213 155.93 to 8 779 438. The question at once presents itself: "What is going to happen in a truss with twice the number of triangles"? These coefficients increase so rapidly, that it is certain they would amount to billions at least. A change of scale might be helpful. At the start the coefficients might be correct to three places of decimals as far as the quarter-point, where all decimals are discarded as far as the half-way point; between the middle and three-quarter points, the last three digits may be neglected, while the remainder of the work may only retain the significant figures up to and including the seventh digit. Now the solution depends on the differences of large numbers, and even if three decimal places were carefully maintained throughout the work, the f-values obtained did not give nearly as fine a check for the $\Sigma M = 0$ equations around the points as some of the other methods. The changes in scale to bring Mao's method within the bounds



— LEGEND —

I - Moment of Inertia in in.^{4}
y - Extreme Fibre Distance in ins.
L - Length of Member in ft.
U - L/y
S - Section Modulus = I/y
P - Primary Stress in lbs/sq.in.
f_{mn} - Secondary Stress in Member mn, at end m, in lbs/sq.in.
$2K_{nn}$ - Change of Angle in Triangle N at Vertex n
ΣM_m - Sum of Moments around Joint m. Positive Moment Counter Clockwise
x - f_{12}
y - f_{21}
Loading: 1500 lbs. at Joint 9.

— FORMULAE —

$K_{nm} = (P_{nq} - P_{mn}) \cdot Cot\,\beta/2$

$K_{nq} = (P_{mn} - P_{nq}) \cdot Cot\,\beta/2 + (P_{m} - P_{mq}) \cdot Cot\,\alpha/2$

$f_{mn} = a_{mn}x + b_{mn}y + k_{mn}$, where a, b & k are Constants

$\tau_{mn} = 2f_{mn} - f_{nm}$

$= \frac{U_{mp}}{U_{mn}} \tau_{mp} + \frac{\Sigma_{mp}^{mn} K_m}{U_{mn}}$, where mp is the Reference Member for Joint m and ΣK is Negative for Top Chord Joints

$\tau_{nm} = 3f_{mn} - 2\tau_{mn}$

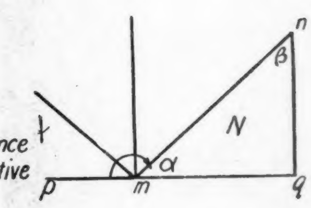
$f_{mn} = \frac{1}{3}\tau_{mn} + \frac{1}{3}(\tau_{mn} + \tau_{nm})$

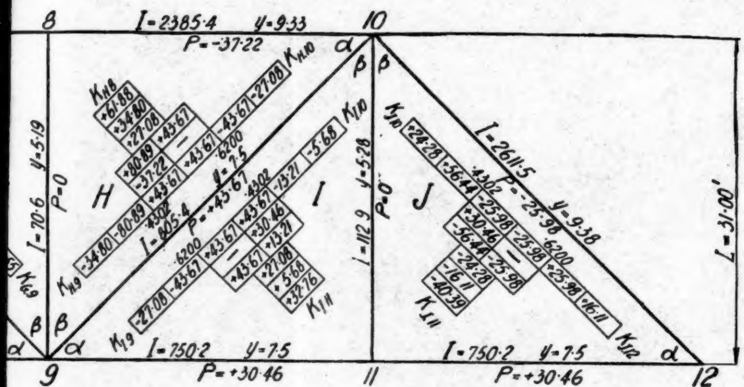
$f_{mn} - f_{nm} = \frac{1}{3}\tau_{mn} - \frac{1}{3}\tau_{nm} = \tau_{mn} - f_{mn}$

$\Sigma M_m = \Sigma S_{mn} \cdot f_{mn} = 0$

Check for Every Triangle N:

$U_{mq}(f_{qm} - f_{mq}) + U_{nq}(f_{nq} - f_{qn}) + U_{mn}(f_{mn} - f_{nm}) = 0.$





SOLUTION OF EQUATIONS

From $\Sigma M_{10} = 0$ & $\Sigma M_{12} = 0$,

$$+3{,}639{,}544x - 137{,}223y = -1{,}406{,}877$$
$$+8{,}579{,}438x - 718{,}960y = -3{,}106{,}013,$$

i.e. $+26\cdot522842x - y = -10\cdot2524868$
$+11\cdot933123x - y = -4\cdot3201472,$

i.e. $+14\cdot589719x = -5\cdot9323396$
$x = -0\cdot40661,$

$y = -4\cdot85214 + 4\cdot32015$
$= -0\cdot53199.$

LAE —

$t\beta/2$

$t\beta/2 + (P_{mn} - P_{mq})\cdot Cot\,\alpha/2$.

$+ k_{mn}$, where

re Constants

$\frac{K_{m}}{U_{mn}}$, where mp is the Reference for Joint m and ΣK is Negative Chord Joints

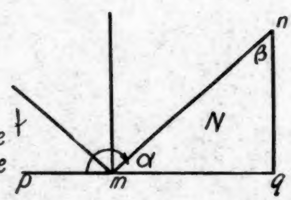
$T_{mn})$

$f_{nm} - r_{mn} - f_{mn}$

$= 0$

Triangle N:

$+ U_{nq}(f_{nq} - f_{qn}) + U_{mn}(f_{mn} - f_{nm}) = 0.$

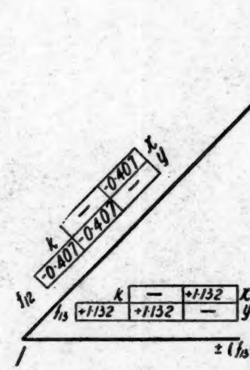


U - 4·246

	x	y	k
r_{21}	-1	+2	0
f_{21}	0	+1	0
$\pm(f_{21}-f_{12})$	∓1	±1	0
f_{12}	+1	0	0
r_{12}	+2	-1	0

S - 278·41

A



U - 3·333

	x	y	k
r_{13}	+2·548	-1·274	+2·415
f_{13}	-2·783	—	—
$\pm(f_{13}-f_{31})$	±5·331	∓1·274	±2·415
f_{31}	-8·114	+1·274	-2·415
r_{31}	-13·445	+2·548	-4·830

S - 100·03

	1-3
K_{81}	+8·05

$\Sigma M_1 = 0$

	S	x	y	k
1-2	278·41	+278·41	—	—
1-3	100·03	-278·41	—	—

$\Sigma M_2 = 0$

	S	x	y	k
2-1	278·41	—	+278·41	—
2-3	21·38	− 64·70	+ 30·92	− 73·50
	Σ	− 64·70	+309·33	− 73·50
2-5	107·39	+1170·55	−106·75	+284·05
2-4	255·67	−1105·85	−202·58	−210·55

ΣK

	2-3	2-5	2-4
K_{A2}	− 12·14		
K_{B2}	+ 2·84	− 9·30	
K_{C2}		+13·53	+ 4·23

$\Sigma M_4 = 0$

	S	x	y	k
4-2	255·67	−1806·56	−1215·20	− 824·79
4-5	13·60	+ 98·34	− 60·34	− 36·96
4-6	255·67	+1708·22	+1275·54	+ 861·75

ΣK

	4-5	4-6
K_{C4}	−30·93	
K_{D4}	+12·84	−18·09

$\Sigma M_6 = 0$

	S	x	y	k
6-4	255·67	+5923·62	+4778·98	+4888
6-5	114·72	+2842·53	+1014·12	+1484
6-7	21·38	− 967·96	+ 364·14	− 301
	Σ	+7798·19	+6157·24	+6071
6-9	114·72	+30479·38	− 2360·02	+10929
6-8	255·67	−38277·57	−3797·22	−17001

ΣK

	6-5	6-7	6-9
K_{D6}	−12·31		
K_{E6}	−18·87	−31·18	
K_{F6}		− 1·79	−32·97
K_{G6}			+12·31

2

	k	−0·824	+1·759	x
f_{24}	+1·356	+2·180	+0·421	y

U = 2·680

	x	y	k	—
τ_{24}	−1·584	+3·169	+1·578	+1·578
f_{24}	−4·325	−0·792	−0·824	
$\pm(f_{24}-f_{42})$	±2·741	±3·961	±2·402	
f_{42}	− 7·066	−4·753	−3·226	
τ_{42}	−9·807	− 8·714	−5·628	

S = 255·67

x	+2·873	+5·402	+2·176	f_{42}
y	+2·529	−3·226	k	

4

	k	+3·371	−2·717	x
f_{46}	−2·000	−5·371	−2·654	y

U = 2·680

	x	y	k	−5·628
τ_{46}	− 9·807	− 8·714	−12·378	−6·750
f_{46}	+ 6·681	+ 4·989	+ 3·371	
$\pm(f_{46}-f_{64})$	∓16·488	∓13·703	∓15·749	
f_{64}	+23·169	+18·692	+19·120	
τ_{64}	+39·657	+32·395	+34·869	

S = 255·67

x	−9·421	−19·365	−0·245	f_{64}
y	−9·944	+19·120	k	

6

f_{68}	+2·279		

f_{21}: x —, −0·532, −0·532; y −0·532, —, k

f_{23}: x +1·230, +0·461, −2·977; y −0·769, −3·438, k

f_{25}: −1·258, k, +2·645, −4·432 x; −3·903, +0·579 y

f_{45}: x −2·940, −0·580, −3·298; y +2·360, −2·718, k

f_{65}: x −10·075, −14·778, −1·838; y −4·703, −12·940, k

f_{67}: x +18·400, +9·348, −4·742; y −9·061, −14·092, k

f_{69}: −1·812, k

C

D

U = 4·246

y	k
+2	0
+1	0
±1	0
0	0
−1	0

S = 278·41

4·320l5

U = 5·871 3U = 17·614

	x	y	k	
f_{23}	−3·026	+1·446	−3·438	—
$\frac{1}{3}\tau_{23}$	−0·241	+0·482	−0·689	−0·689
$\frac{1}{3}(\tau_{23}+\tau_{32})$	−2·785	+0·964	−2·749	−0·914
$\frac{1}{3}\tau_{32}$	−2·544	+0·482	−2·060	−1·146
$(f_{23}-f_{32})$	+2·303	—	+1·371	
f_{32}	−5·329	+1·446	−4·809	

Check		x	y	k
	2-3	+13·521	—	+8·049
	1-3	−17·768	+4·246	−8·049
	1-2	+4·246	−4·246	—

S = 21·38

U = 5·310 3U = 15·930

	x	y	k	
f_{25}	+10·900	− 0·994	+2·645	—
$\frac{1}{3}\tau_{25}$	− 0·267	+ 0·533	−0·584	−0·584
$\frac{1}{3}(\tau_{25}+\tau_{52})$	+11·167	− 1·527	+3·229	+4·662
$\frac{1}{3}\tau_{52}$	+11·434	−2·060	+3·813	−0·849
$(f_{25}-f_{52})$	− 11·701	+2·593	−4·397	
f_{52}	+22·601	−3·587	+7·042	

	x	y	k
2-5	−62·132	+13·769	−23·348
3-5	+75·652	−13·769	+31·394
2-3	−13·521	—	− 8·049

S = 107·39

U = 5·973 3U = 17·919

	x	y	k	
f_{45}	+7·231	−4·437	−2·718	−0·842
$\frac{1}{3}\tau_{45}$	−1·467	−1·303	−2·568	−1·726
$\frac{1}{3}(\tau_{45}+\tau_{54})$	+8·698	−3·134	−0·150	+4·144
$\frac{1}{3}\tau_{54}$	+10·165	−1·831	+2·418	−1·726
$(f_{45}-f_{54})$	− 11·632	+0·528	−4·986	
f_{54}	+18·863	−4·965	+2·268	

	x	y	k
4-5	−69·478	+3·154	−29·781
2-5	+62·132	−13·769	+23·348
2-4	+ 7·346	+10·615	+ 6·437

S = 13·60

U = 5·310 3U = 15·930

	x	y	k	
f_{65}	+ 24·778	+8·840	+12·940	+5·866
$\frac{1}{3}\tau_{65}$	+ 6·672	+5·450	+ 5·093	−0·773
$\frac{1}{3}(\tau_{65}+\tau_{56})$	+18·106	+3·390	+ 7·847	+4·662
$\frac{1}{3}\tau_{56}$	+ 11·434	−2·060	+2·754	−1·908
$(f_{65}-f_{56})$	− 4·762	+7·510	+2·339	
f_{56}	+29·540	+1·330	+10·601	

	x	y	k
6-5	−25·286	+39·878	+12·420
4-5	+69·478	− 3·154	+29·781
4-6	−44·188	−36·724	−42·207

S = 114·72

U = 5·871 3U = 17·614

	x	y	k
f_{67}	− 45·274	+17·032	−14·090
$\frac{1}{3}\tau_{67}$	+ 6·034	+ 4·929	+ 3·535
$\frac{1}{3}(\tau_{67}+\tau_{76})$	−51·308	+12·103	−17·625
$\frac{1}{3}\tau_{76}$	− 57·342	+ 7·174	−21·160
$(f_{67}-f_{76})$	+ 63·376	− 2·245	+24·695
f_{76}	−108·650	+19·277	−38·785

	x	y	k
6-7	+372·080	−13·180	+144·984
5-7	−397·390	+53·058	−32·570
5-6	+ 25·286	−39·878	−12·420

S = 21·38

A

B

E

f_{32}: x +2·167, −4·809, −3·411 k; y −0·769, +1·398

f_{52}: x −9·190, −7·282, −0·240; y +1·908, +7·042, k

f_{54}: x −7·670, +2·268, −2·761 k; y +2·641, −5·029

f_{56}: k −2·118, +10·601, −12·011 x; −12·719, −0·708 y

f_{76}: x +44·178, −38·785, −4·862 k; y −10·255, +33·923

U = 3·333

y	k	
−1·274	+2·415	+2·415
—	—	
∓1·274	±2·415	
+1·274	−2·415	
+2·548	−4·830	

S = 100·03

3

x	+3·299	+2·621	+0·206	f_{31}
y	−0·678	−2·415	k	

	k	+3·443	−3·762	x
f_{35}	+0·523	−2·920	+0·842	y

U = 3·333

	x	y	k	−4·830
τ_{35}	−13·445	+2·548	−5·976	−1·146
f_{35}	+ 9·253	−1·583	+3·443	
$\pm(f_{35}-f_{53})$	∓22·698	±4·131	∓9·419	
f_{53}	+31·951	−5·714	+12·862	
τ_{53}	+54·649	−9·845	+22·281	

S = 100·03

ΣK

	3-2	3-5
K_{A3}	−20·19	
K_{B3}	+16·37	−3·82

$\Sigma M_3 = 0$

	S	x	y	k
3-1	100·03	−811·64	+127·44	−241·57
3-2	21·38	−113·93	+ 30·92	−102·82
3-5	100·03	+925·57	−158·36	+344·39

5

x	−12·992	−9·952	+2·910	f_{53}
y	+3·040	+12·862	k	

	k	−22·922	+26·259	x
f_{57}	+0·106	+23·028	−3·231	y

U = 3·333

	x	y	k	+22·281
τ_{57}	+ 54·649	− 9·845	+16·853	− 5·428
f_{57}	− 64·580	+ 6·074	− 22·922	
$\pm(f_{57}-f_{75})$	±119·229	∓15·919	± 39·775	
f_{75}	−183·809	+ 21·993	− 62·697	
τ_{75}	−303·038	+37·912	−102·472	

S = 143·52

ΣK

	5-2	5-4	5-6	5-7
K_{B5}	−13·53			
K_{C5}	−17·40	−30·93		
K_{D5}		+ 0·53	−30·40	
K_{E5}			+12·31	−18·09

$\Sigma M_5 = 0$

	S	x	y	k
5-3	100·03	+3196·06	−571·57	+1286·59
5-2	107·39	+2427·12	−385·21	+ 756·24
5-4	13·60	+ 256·54	− 67·52	+ 30·84
	Σ	+5879·72	−1024·30	+2073·67
5-6	114·72	+3388·83	+ 152·58	+1216·15
5-7	143·52	−9268·55	+ 871·72	−3289·82

7

x	+74·739	+63·039	+0·342	f_{75}
y	−11·700	−62·697	k	

f_{79}	+0·383		

ΣK

	7-6	7-9
K_{E7}	−31·18	
K_{F7}	+10·52	−20·66

$\Sigma M_7 = 0$

	S	x	y	k
7-5	143·52	−26380·27	+3156·44	−8998
7-6	21·38	− 2322·94	+ 412·14	− 829
7-9	143·52	+28703·21	−3568·58	+9827

k
74·79
36·96
51·75

$\Sigma M_6 = 0$

	S	x	y	k
6-4	255·67	+5923·62	+4178·98	+4888·41
6-5	114·72	+2842·53	+1014·12	+1484·48
6-7	21·38	- 967·96	+ 364·14	- 301·24
	Σ	+7798·19	+6157·24	+6071·65
6-9	114·72	+30479·38	- 2360·02	+10929·83
6-8	255·67	-38277·57	- 3797·22	-17001·48

ΣK

	6-5	6-7	6-9	6-8
K_{D6}	-12·31			
K_{E6}	-18·87	-31·18		
K_{F6}		- 1·79	-32·97	
K_{G6}			+12·31	-20·66

$\Sigma M_8 = 0$

	S	x	y
8-6	255·67	-86694·37	-15876·85
8-9	13·60	+ 901·10	- 825·33
8-10	235·67	+85793·27	+16702·18

ΣK

	8-9	8-1
K_{G8}	-36·86	
K_{H8}	-61·88	-98·7

+3·371	-2·717	x
-5·371	-2·654	y

U-2·680

	x	y	k	
t_{46}	- 9·807	- 8·714	-12·378	-5·628
f_{46}	+ 6·681	+ 4·989	+ 3·371	-6·750
$\pm(f_{46}-f_{64})$	∓16·488	∓13·703	∓15·749	
f_{64}	+23·169	+18·692	+19·120	
t_{64}	+39·657	+32·395	+34·869	

S-255·67

6

x	-9·421	-19·365	-0·245	f_{64}
y	-9·944	+19·120	k	

	k	-66·498	+60·876	x
f_{68}	+2·279	+68·777	+ 7·901	y

x	-10·075	-14·778	-1·838	f_{65}
y	-4·703	+12·940	k	

x	+18·409	+9·348	-4·742	f_{67}
y	-9·061	-14·092	k	

f_{69}	-1·812	k	+95·274	+108·030 x
		-97·086	+ 10·944	y

U-2·680

	x	y	k	
t_{68}	+ 39·657	+ 32·395	+ 27·160	+34·869
f_{68}	-149·715	- 14·852	- 66·498	- 7·709
$\pm(f_{68}-f_{86})$	±189·372	± 47·247	± 93·658	
f_{86}	-339·087	- 62·099	-160·156	
t_{86}	-528·459	-109·346	-253·814	

S-255·67

8

x	+137·877	+170·913	+10·757	f_{86}
y	+33·036	-160·156	k	

f_{810} -10·

x	-26·941	+5·343	+·886	f_{89}
y	+32·284	-3·457	k	

D

G

-0·842
-1·726
+4·144
-1·726

U=5·310 3U=15·930

	x	y	k	
f_{65}	+ 24·778	+8·840	+12·940	+5·866
$\frac{1}{3}t_{65}$	+ 6·672	+5·450	+ 5·093	-0·773
$\frac{1}{3}(t_{65}+t_{56})$	+18·106	+3·390	+ 7·847	+4·662
$\frac{1}{3}t_{56}$	+ 11·434	- 2·060	+ 2·754	-1·908
$(f_{65}-f_{56})$	- 4·762	+7·510	+ 2·339	
f_{56}	+29·540	+ 1·330	+10·601	

	x	y	k
6-5	+25·286	+39·878	+12·420
4-5	+69·478	- 3·154	+29·781
4-6	-44·188	-36·724	-42·207

S-114·72

U=5·871 3U-17·614

	x	y	k	
f_{67}	- 45·274	+17·032	-14·090	+5·305
$\frac{1}{3}t_{67}$	+ 6·034	+ 4·929	+ 3·535	-1·770
$\frac{1}{3}(t_{67}+t_{76})$	- 51·308	+12·103	-17·625	-19·390
$\frac{1}{3}t_{76}$	- 57·342	+ 7·174	- 21·160	- 1·770
$(f_{67}-f_{76})$	+ 63·376	- 2·245	+24·695	
f_{76}	-108·650	+19·277	-38·785	

	x	y	k
6-7	+372·080	-13·180	+144·984
5-7	-397·390	+53·058	-32·570
5-6	+ 25·286	-39·878	- 12·420

S-21·38

U=5·310 3U=15·930

	x	y	k	
f_{69}	+265·685	-20·572	+95·274	+5·866
$\frac{1}{3}t_{69}$	+ 6·672	+ 5·450	+ 3·796	-2·070
$\frac{1}{3}(t_{69}+t_{96})$	+259·013	-26·022	+91·478	+88·455
$\frac{1}{3}t_{96}$	+252·341	-31·472	+87·682	- 0·773
$(f_{69}-f_{96})$	-245·669	+36·922	-83·886	
f_{96}	+511·354	- 57·494	+179·160	

	x	y	k
6-9	-1304·502	+196·056	-445·435
7-9	+1676·606	-209·236	+590·428
7-6	- 372·080	+ 13·180	-144·984

S-114·72

U=5·973 3U-17·91

	x	y	k
f_{89}	+ 66·257	- 60·686	- 3·45
$\frac{1}{3}t_{89}$	- 79·037	- 16·354	-40·0
$\frac{1}{3}(t_{89}+t_{98})$	+145·294	- 44·332	+36·5
$\frac{1}{3}t_{98}$	+224·331	- 27·978	+76·5
$(f_{89}-f_{98})$	-303·368	+ 11·624	-116·5
f_{98}	+369·625	- 72·310	+113·1

	x	y	k
8-9	-1812·017	+ 69·430	-696·
6-9	+1304·502	-196·056	+445·
6-8	+ 507·517	+126·622	+251·

S- 13·60

E

F

2·011	x
-0·708	y

k	-38·785	-44·178	x	
	-4·862	+33·923	-10·255	y
f_{76}				

	k	+68·475	-81·320	x
f_{79}	+0·383	-68·092	+13·228	y

x	-207·922	-177·336	+1·824	f_{96}
y	+30·586	+179·160	k	

f_{98}	+1·314	+115·140	-150·294	x
	k	-111·826	+36·468	y

f_{910} +1·665 k +346·33 -350

f_{911} -9·

k	-22·922	+26·259	x
6	+23·028	-3·231	y

U-3·333

	x	y	k	
t_{57}	+ 54·649	- 9·845	+16·853	+22·281
f_{57}	- 64·580	+ 6·074	- 22·922	- 5·428
$\pm(f_{57}-f_{75})$	∓119·229	±15·919	± 39·775	
f_{75}	-183·809	+ 21·993	- 62·697	
t_{75}	-303·038	+37·912	-102·472	

S=143·52

x	+74·739	+63·039	+0·342	f_{75}
y	-11·700	-62·697	k	

7

U-3·333

	x	y	k	
t_{79}	- 303·038	+ 37·912	-108·671	-102·472
f_{79}	+199·994	- 24·865	+ 68·475	- 6·199
$\pm(f_{79}-f_{97})$	∓503·032	± 62·777	∓177·146	
f_{97}	+ 703·026	- 87·642	+245·621	
t_{97}	+1206·058	-150·419	+422·767	

S-143·52

x	-285·858	-239·233	+6·388	f_{97}
y	+46·625	+245·621	k	

9

5-7
-18·09

ΣK

	7-6	7-9
K_{E7}	-31·18	
K_{F7}	+10·52	-20·66

ΣK

	9-6	9-8	9-1
K_{F9}	-12·31		
K_{G9}	-24·55	-36·86	
K_{H9}		-34·80	- 71·
K_{I9}			- 27·

k
286·59
756·24
30·84
073·67
216·15
289·82

$\Sigma M_7 = 0$

	S	x	y	k
7-5	143·52	-26380·27	+3156·44	-8998·27
7-6	21·38	- 2322·94	+ 412·14	- 829·22
7-9	143·52	+28703·21	-3568·58	+9827·49

$\Sigma M_9 = 0$

	S	x	y
9-7	143·52	-100898·29	-12578·38
9-6	114·72	+ 58662·53	- 6595·71
9-8	13·60	+ 5026·90	- 983·42
	Σ	-16458·72	-20157·51
9-10	107·39	+91480·71	+ 732·29
9-11	100·03	-256068·43	+19425·22

SECONDARY STRESSES BY MAO'S ANALYTICAL METHOD

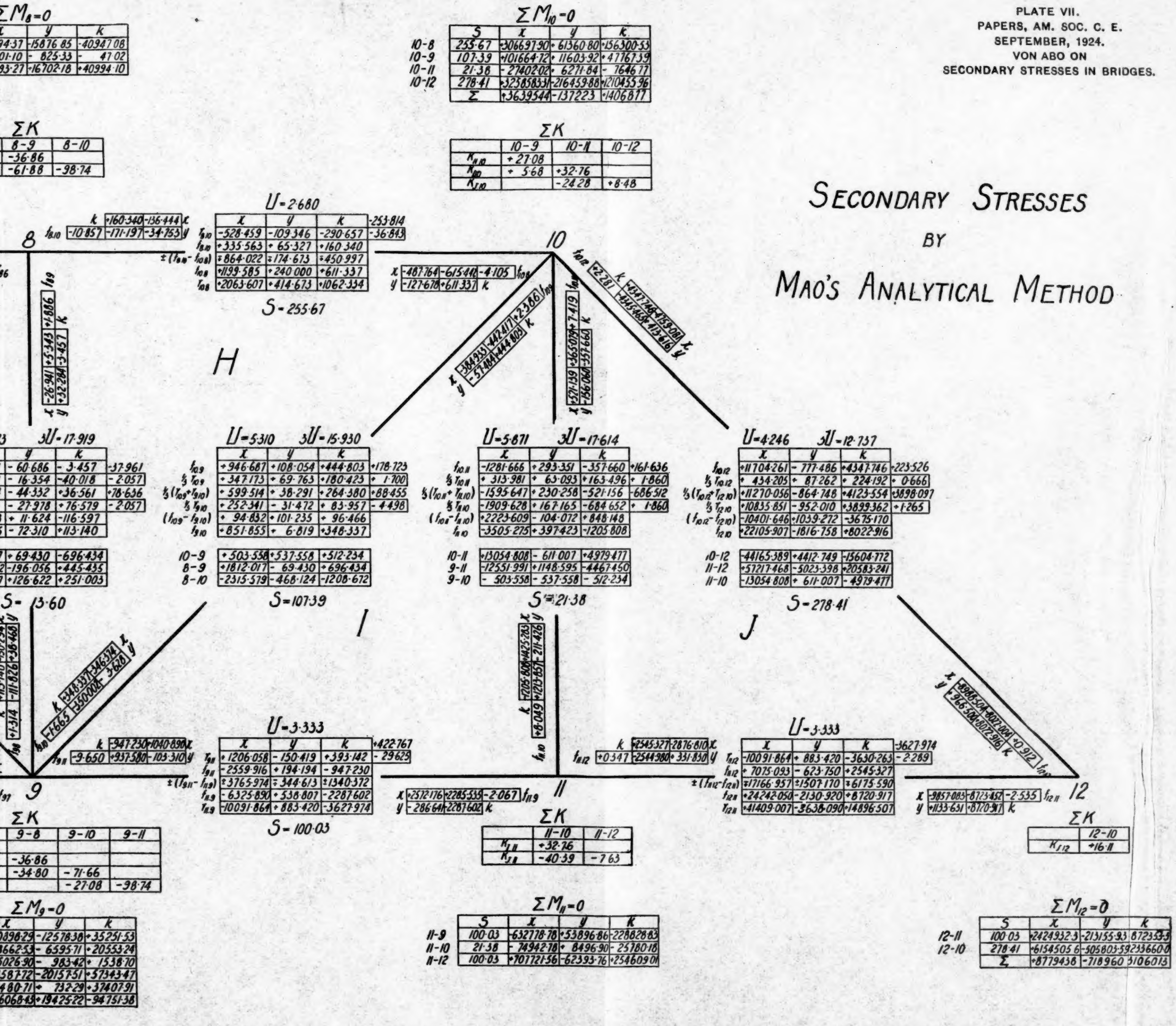


TABLE 24.—COMPARISON OF SECONDARY STRESSES DERIVED BY VARIOUS METHODS, TRUSS OF PLATE VI; LOADING, 1 500 LB. AT JOINT (9).

Joint.	Member.	Primary stress, in pounds per square inch.	SECONDARY STRESS, IN POUNDS PER SQUARE INCH.								
			Manderla.	Manderla-Gauss.	Muller-Breslau.	Ritter.	Mohr (Semi-Graphical).	Mohr (Elastic Weight). 5 Decimal Places.	Mohr (Elastic Weight). 10 Decimal Places.	Mao Analytical.	Mao Graphical.
1	1-2	−12.99	− 0.41	0.41	0.41	0.41	0.41	0.41	0.41	0.41	0.42
			+ 0.42	0.41	0.41	0.42	0.42	0.41	0.41	0.41	0.43
	1-3	+15.23	± 1.12	1.13	1.14	1.16	1.14	1.13	1.13	1.13	1.15
2	2-4	−18.61	+ 1.35	1.38	1.40	1.36	1.38	1.38	1.38	1.36	1.40
			− 1.38	1.41	1.43	1.39	1.41	1.41	1.41	1.39	1.43
	2-5	+21.83	∓ 1.26	1.25	1.26	1.25	1.25	1.25	1.25	1.26	1.20
	2-3	....	∓ 2.89	2.88	2.90	2.90	2.88	2.89	2.89	2.98	2.93
	2-1	−12.99	− 0.58	0.57	0·58	0.57	0.58	0.57	0.57	0.54	0.59
			+ 0.57	0.56	0.57	0.57	0.57	0.56	0.56	0.53	0.58
3	3-1	+15.23	± 0.16	0.17	0.18	0·25	0.18	0.17	0.17	0.21	0.15
	3-2	....	∓ 3.33	3.32	3.33	3.32	3.31	3.32	3.32	3.41	3.37
	3-5	+15.23	± 0.53	0.56	0.54	0.58	0.54	0.54	0.54	0.52	0.23
4	4-6	−18.61	− 1.95	1.92	1.83	1.88	1.92	1.91	1.91	2.00	1.98
			+ 1.99	1.96	1.87	1.98	1.96	1.96	1.95	2.04	2.02
	4-5	....	∓ 3.22	3.21	3.16	3.21	3.21	3.20	3.21	3.30	3.23
	4-2	−18.61	+ 2.10	2.13	2.22	2.15	2.13	2.13	2.13	2.23	2.15
			− 2.05	2.08	2.17	2.10	2.08	2.08	2.08	2.18	2.10
5	5-3	+15.23	± 2.82	2.83	2.83	2.89	2.82	2.83	2.83	2.91	2.77
	5-2	+21.83	∓ 0.30	0.29	0.30	0.28	0.29	0.29	0.29	0.24	0.20
	5-4	....	∓ 2.69	2.70	2.64	2.66	2.67	2.67	2.67	2.76	2.72
	5-6	−19.85	∓ 2.09	2.08	2.08	2.07	2.07	2.08	2.07	2.12	2.10
	5-7	+24.02	± 0.15	0.16	0.16	0.22	0.16	0.16	0.15	0.11	0.28
6	6-8	−37.22	+ 2.27	2.28	2.28	2.31	2.28	2.29	2.26	2.28	2.41
			− 2.32	2.34	2.33	2.37	2.33	2.34	2.31	2.33	2.46
	6-9	+19.85	∓ 1.80	1.80	1·80	1.79	1.79	1.78	1.77	1.81	1.70
	6-7	....	∓ 4.60	4.60	4.62	4.62	4.60	4.61	4.60	4.74	4.58
	6-5	−19.85	∓ 1.86	1.85	1.86	1.85	1.84	1.85	1.84	1.84	1.90
	6-4	−18.61	− 0.31	0.27	0.25	0.25	0.27	0.28	0.26	0.25	0.38
			+ 0.30	0.27	0.25	0.24	0.27	0.28	0.25	0.25	0.37

TABLE 24.—(*Continued*).

Joint.	Member.	Primary stress, in pounds per square inch.	Secondary Stress, in Pounds per Square Inch.								
			Manderla.	Manderla-Gauss.	Muller-Breslau.	Ritter.	Mohr (Semi-Graphical).	Mohr (Elastic Weight).		Mao Analytical.	Mao Graphical.
								5 Decimal Places.	10 Decimal Places.		
7	7-5	+24.02	± 0.30	0.31	0.30	0.32	0.31	0.30	0.29	0.34	0.42
	7-6	....	∓ 4.73	4.72	4.74	4.74	4.73	4.74	4.73	4.86	4.67
	7-9	+24.02	± 0.40	0.39	0.40	0.48	0.39	0.41	0.41	0.38	0.25
8	8-10	−37.22	−10.52	10.52	10.55	10.55	10.52	10.53	10.58	10.86	10.62
			+10.75	10.75	10.79	10.79	10.75	10.77	10.82	11.09	10.85
	8-9	....	± 1.85	1.85	1.88	1.84	1.86	1.88	1.82	1.89	1.92
	8-6	−37.22	+10.64	10.65	10.69	10.64	10.65	10.67	10.72	10.99	10.76
			−10.41	10.42	10.45	10.41	10.42	10.43	10.49	10.76	10.53
9	9-7	+24.02	± 6.15	6.14	6.18	6.19	6·15	6.21	6.14	6.39	6.10
	9-6	+19.85	± 1.68	1.68	1.69	1.69	1.69	1.71	1.62	1.82	1.80
	9-8	....	± 1.30	1.29	1.32	1.29	1.29	1.33	1.29	1.31	1.38
	9-10	+43.67	∓ 1.47	1.47	1.49	1.46	1.47	1.47	1.44	1.66	1.45
	9-11	+30.46	∓ 9.32	9.33	9.39	9.33	9.33	9.48	9.29	9.65	9.27
10	10-12	−25.98	+ 2.30	2.31	2.43	2.34	2.31	2.46	2.33	2.28	2.18
			− 2.33	2.34	2.44	2.37	2.34	2.48	2.35	2.31	2.21
	10-11	....	± 7.19	7.19	7.26	7.23	7.21	7.29	7.20	7.42	7.47
	10-9	+43.67	± 2.39	2.39	2.42	2.40	2.38	2.32	2.41	2.39	2.40
	10-8	− 37.22	− 4.22	4.21	4.36	4.24	4.22	4.35	4.24	4.20	4.32
			+ 4.13	4.12	4.27	4.14	4.12	4.26	4.15	4.10	4.23
11	11-9	+30.46	∓ 2.05	2.05	2.07	2.04	2.04	1.94	2.04	2.07	2.08
	11-10	....	± 7.83	7.83	7.88	7.89	7.86	7.85	7.84	8.05	7.98
	11-12	+30.46	± 0.37	0.38	0.38	0.40	0.36	0.26	0.36	0.35	0.35
12	12-11	+30.46	∓ 2.54	2.54	2.55	2.55	2.55	2.64	2.55	2.53	2.65
	12-10	−25.98	+ 0.92	0.92	0.94	0.92	0.93	0.96	0.93	0.92	0.78
			− 0.91	0.91	0.93	0.91	0.92	0.95	0.92	0.91	0.77

of practice in solving the secondary stresses for a truss with only a few more triangles than that used in Plate VII, are, therefore, certain to introduce large discrepancies.

Dr. Mao claims that the method is self-checking, that is, that if a computer makes an error, he is almost immediately aware of it. The writer's experience proved the contrary. An error which he made near the beginning of his computations was not discovered until a considerable time afterward and then only by comparing the results with those of the other methods. If there had been no other solution, the error would only have been detected when checking the $\Sigma M = 0$ equations.

This problem concerns a Warren truss with triangular elements. Suppose it had been a truss with subdivided panels, or even the top lateral system of the bridge, shown in Fig. 50. In the former case, some elements other than triangles appear, while all the triangles of the latter case have a vertex in the interior, instead of on the boundary of the frame. To adopt Mao's method to such cases would make an already weak method of smaller value still.

Mao's Graphical Method.—In this case, too, the unsymmetrical loading brings out certain features that were not so pronounced in the symmetrical case treated by Dr. Mao. It must be remembered that Figs. 56 and 57 give only those lines and points that are essential in the final solution. In carrying out the solution, the number of lines that must be drawn is enormous. Consider the problem of finding the base line for Member 5-7. There are required at least three points that must prove to be collinear, before the required line is accurately determined.

An explanation will be given of the steps necessary to arrive at one of these three points. (The process is repeated in its entirety for each of the other two, while each of the many base and vertex lines call for similar determinations.) At the stage where the base line for Member 5-7 is the next line to be determined, the base and vertex lines for the triangles, *A, B, C,* and *D,* have already been determined.

First, assume the reference contour for Δ *E,* namely, r_{64}, to be zero, and with this value finish a portion of the deformation contour to obtain the corresponding value for r_{65}.

Next, assume some value for r_{56}, and proceed to complete the corresponding deformation contour for Joint (5). Now, draw the vertical line through the end of r_{53} to cut the vertex line of Δ *D,* and also that of Δ *C,* as r_{53} is the reference contour for both these triangles. Through each of the two points thus found, draw a parallel to the base line of the corresponding triangle. Returning to the deformation contour at Joint (6), draw a line at 45° from the point, (6) (it being also the end of r_{64} as r_{64} is zero), until it meets the parallel to the base line of Δ *D.* From this point a line downward at 45° to meet Member 4-6, gives the value of r_{46}. To complete the corresponding deformation contour round Joint (4), draw a line from the end of r_{42}, at 45° upward to the left to cut the parallel to the base line of Δ *C,* and from this point, a line downward to cut Member 2-4, giving the value of r_{24}. With this the deformation contour around Joint (2) may be completed.

The work is now at the point where the value of the reference contour for ΔB, namely, r_{21}, is known. Draw a line vertically downward through the end of r_{21} to cut the vertex line of ΔB; through this point draw a line parallel to the base line of ΔB. Returning to the deformation contour of Joint (5), draw a line through the end of r_{53} downward at 45° to cut the parallel just found, and from the point of intersection draw a line at 45° upward to the left to cut Member 3-5, giving r_{35}.

This gives all the r's of all the members entering Joint (5) except r_{75}. Making use of the fact that $3 f_{5n} = 2 r_{5n} + r_{n5}$, determine the points on Members 5-3, 5-2, 5-4, and 5-6, that are distant $3 f_{53}$, $3 f_{52}$, $3 f_{54}$, and $3 f_{56}$, respectively, from Joint (5). Draw the S's through these points and complete the equilibrium polygon. S_{57} is regarded as the resultant of R_5, S_{53}, S_{52}, S_{54}, and S_{56}. Its distance from Joint (5) along Member 5-7 is $3 f_{57}$.

Now, $r_{75} = 3 f_{57} - 2 r_{57}$, and as r_{57} was assumed at the start, the value of r_{75} is known. Draw a line through the end of r_{57} downward to the right and at 45°; draw another line through the end of r_{75} downward to the left and at 45 degrees. The point of intersection is the first point on the required base line.

It is at once evident that the amount of labor is prodigious, while the likelihood of error in the process may be the cause of many hours being wasted. Of course, where two or three members enter a joint, the process of determining the base and vertex lines, although complicated, is not as extended as that just explained.

In applying this graphical method, the number of lines becomes so great that the draftsman becomes lost. The natural thing to do would be to erase those lines that had been used to find a point and would not be used again. No erasures should be made, however, until the base line or the vertex line has been definitely found, otherwise the three points may be non-collinear and the lines may be lost from which these points were found; this would necessitate starting the problem of finding that particular base or vertex line from the very beginning.

There is no doubt that if all the checks are carefully carried out, Mao's graphical method will be correct, as the drawing proceeds, and will give results that are surprisingly accurate. However, it is not to be recommended. In the first place, it requires a large scale for reasonable accuracy; the result is an unwieldy drawing. Again, it requires ingenuity to adapt it to the solution of trusses with subdivided panels, for, as it is now, it cannot be so used. Thirdly, the labor involved to arrive at the "base" and "vertex" lines is excessive, and because of the many repetitions of the same process, is bound to become extremely monotonous. Finally, any graphical method is objectionable in the drafting-room, as a complete and thorough check by a second man requires an entirely new solution of the whole problem.

Ritter's Graphical Method.—This method entails much more than appears in Fig. 53. In this case, as in Mao's graphical method, the great number of lines that were required to arrive at the final solution shown, have been erased. The method gives reasonably accurate results. Bridge engineers, however,

would prefer an analytical method, which can be carried to any degree of accuracy. It may be said in favor of Ritter's method, that it may be checked by drawing a fresh set of equilibrium polygons. The writer, however, has an objection that eliminates Ritter's method. When the influence lines for the secondary stresses are required, and several conditions of loading are analyzed, Ritter's method cannot be used as advantageously as that of Manderla.

Müller-Breslau's and Mohr's "Elastic Weight" Methods.—Both these methods are to be discarded for the simple reason that the derivation of the (4n-6) unknown quantities from a set of (4n-6) simultaneous equations requires many places of decimals for reasonably accurate results. During the process of elimination, errors are likely to enter and cannot be detected until the $\Sigma M = 0$ checks for the joints are performed. Engineers have been bitterly opposed to methods that do not contain some checking features; how any one, therefore, can advocate these methods is difficult to understand.

Mohr's Semi-Graphical Method.—This method is better than all the preceding methods, when the solution of the simultaneous equations is performed by the method of Gauss' normal equations. Its accuracy depends on the accuracy with which the Williot diagram is drawn. Errors are liable to enter in the determination of the signs of the ρ's in the diagram, but, as the diagram is clear and contains few lines, it may easily be checked independently. It has the advantage over Manderla's method in that the changes of slope of the various members are at once found and, therefore, can be readily tested in practice.

Manderla's Method.—On comparison with the standard (Mohr's elastic weight method, carried to ten places of decimals) Mohr's semi-graphical and Manderla's methods give results that are close (Table 24). The calculations of these methods involve only three decimal places. In the matter of checking the computations, analytical methods are generally preferred to graphical. The writer personally much prefers Manderla's method to that of Mohr and has found that it could be used advantageously in the determination of influence lines.

In the case of plane frames that are not entirely composed of triangular elements Manderla's method shows itself more advantageous than any other. There is one exception with rectangular frames that suffer bending but no change in the length of their members, Mohr's elastic weight method may be used to advantage.

Although Manderla's method still retains the features that made scientists anxious to improve on it, the writer's investigations have convinced him that none of the later ones is superior. There is a point of great importance that seems to be overlooked by many; the very fact that Manderla was able to obtain a set of relations between a portion of the unknown quantities (the number being the smallest possible, namely, the number of the joints) counts heavily in favor of that method.

The writer would urge that Gauss' theory of normal equations, or Turneaure's method of elimination, be used in every solution. With regard to the former, the instructions are put in such a manner that an inexperienced

engineer can carry out the solution of the unknown τ's to any desired degree of accuracy by means of a calculating machine; if he assures himself of the check at every stage of the solution, he cannot go wrong. The method is superior to that by substitution, where many trials are performed before any reasonable result is found, and where the additions and subtractions that are continually carried out on papers at the side are most annoying. The very directness of Gauss' method, where the object is definitely in view, ought to appeal to every one. Finally, the process of multiplying the equations by quantities far less than unity never allows the coefficients to become large. The solution may be made correct to five decimal places almost as easily as to three.

PART IV.—SOLUTION OF SECONDARY STRESSES BY INFLUENCE LINES

1.—Introduction

The stresses that occur in a bridge, subjected to (1) dead load and (2) dead load plus live load, will now be investigated. The bridge shown in Plate VI and Figs. 49 and 50 will again be used. As symmetry is maintained about the longitudinal central vertical plane under both conditions of loading, it will be possible to use the same set of influence lines in each case. This set will tell when a certain member is taxed most heavily, what fiber suffers most, and what is the value of the stress in it at the instant.

To make the problem complete, it would be necessary to analyze the bridge for wind load and traction also, and combine the resulting solutions with those found in this paper to get the absolute maximum.

A certain amount of work has been omitted in order to save space. For example, only the equations in Section 2 resulting from 1 000 lb. at Panel Point (7) and 3 000 lb. at Panel Point (11), are reproduced, the work of obtaining these being exactly similar to that found in Tables 1 to 5, inclusive.

2.—Influence Lines for the Bridge Trusses

Load of 1 Kip at Panel Point (7).—Tables are prepared corresponding to Tables 2, 3, and 4, for use in forming the equations. As there is symmetry about the central vertical member, 6-7, no bending moments are found in it, while certain relations between the τ's reduce the number of equations to be solved to five. If the equations for Joints (1), (2), (3), (4), and (5) are written, it will be found that τ_6 and τ_7 enter into the last two equations. As there can be no bending in Member 6-7,

$$\tau_{67} = \tau_{76} = 0$$

Table 3 gives:

$$\tau_{67} = \tau_6 + 24.61 + 35.38$$

and,

$$\tau_{76} = \tau_7 - 110.92$$

whence,

$$\tau_6 = 59.99 \quad \text{and} \quad \tau_7 = +110.92$$

Finally, substituting these values in Equations (4) and (5) the equations for joints:

Joint (1)... $+15.930\,\tau_1 + 5.464\,\tau_2 + 2.501\,\tau_3 + 27.96 = 0$

Joint (2)... $+30.806\,\tau_2 + 7.951\,\tau_4 + 1.685\,\tau_5 + 0.303\,\tau_3 + 5.464\,\tau_1 - 2.58 = 0$

Joint (3)... $+10.610\,\tau_3 + 2.501\,\tau_1 + 0.303\,\tau_2 + 2.501\,\tau_5 - 25.50 = 0$

Joint (4)... $+32.184\,\tau_4 + 0.190\,\tau_5 + 7.951\,\tau_2 + 997.68 = 0$

Joint (5)... $+19.528\,\tau_5 + 2.501\,\tau_3 + 1.685\,\tau_2 + 0.190\,\tau_4 + 190.82 = 0$

Load of 1½ Kips at Panel Point (9).—This is the work completed on Fig. 51 and Tables 1 to 7.

Load of 3 Kips at Panel Point (11).—A process similar to that followed in the preceding case gives:

Joint (1)... $+15.930\,\tau_1 + 5.464\,\tau_2 + 2.501\,\tau_3 + 27.96 = 0$

Joint (2)... $+30.806\,\tau_2 + 7.951\,\tau_4 + 1.685\,\tau_5 + 0.303\,\tau_3 + 5.464\,\tau_1 - 2.58 = 0$

Joint (3)... $+10.610\,\tau_3 + 2.501\,\tau_1 + 0.303\,\tau_2 + 2.501\,\tau_5 - 25.50 = 0$

Joint (4)... $+32.184\,\tau_4 + 7.951\,\tau_6 + 0.190\,\tau_5 + 7.951\,\tau_2 + 828.72 = 0$

Joint (5)... $+19.528\,\tau_5 + 2.501\,\tau_3 + 1.685\,\tau_2 + 0.190\,\tau_4 + 1.800\,\tau_6 + 3.588\,\tau_7 - 610.88 = 0$

Joint (6)... $+39.610\,\tau_6 + 7.951\,\tau_8 + 1.800\,\tau_9 + 0.303\,\tau_7 + 1.800\,\tau_5 + 7.951\tau_4 + 842.42 = 0$

Joint (7)... $+14.958\,\tau_7 + 3.588\,\tau_5 + 0.303\,\tau_6 + 3.588\,\tau_9 - 443.04 = 0$

Joint (8)... $+32.184\,\tau_8 + 7.951\,\tau_{10} + 0.190\,\tau_9 + 7.951\,\tau_6 + 2\,619.39 = 0$

Joint (9)... $+19.528\,\tau_9 + 3.588\,\tau_7 + 1.800\,\tau_6 + 0.190\,\tau_8 + 1.685\,\tau_{10} + 2.501\,\tau_{11} + 227.60 = 0$

Joint (10).. $+30.806\,\tau_{10} + 5.464\,\tau_{12} + 0.303\,\tau_{11} + 1.685\,\tau_9 + 7.951\,\tau_8 + 5271.52 = 0$

Joint (11).. $+10.610\,\tau_{11} + 2.501\,\tau_9 + 0.303\,\tau_{10} + 2.501\,\tau_{12} - 2\,911.27 = 0$

Joint (12).. $+15.930\,\tau_{12} + 2.501\,\tau_{11} + 5.464\,\tau_{10} + 1\,034.14 = 0$

Simultaneous Equations.—It will be noticed that the first three equations of all the sets are identical, while the equations for Joints (1), (2), (3), (4), (5), and (7) are the same for the last two sets. These facts are used in the solutions, Table 25. The blank spaces could be filled with the quantities in the corresponding spaces of the preceding cases; it is evident that a certain amount of work for Case II was already done in solving Case I, while an even larger amount is duplicated in Cases II and III. The solution of such sets of equations is exceedingly easy and once all the coefficients in the τ-columns are found, any extra solution offers little trouble. The checks that occur all through the work make it impossible for a computer to go wrong in that part of the problem.

TABLE 25.—SOLUTION OF SIMULTANEOUS

Number of equation.	Remarks.	τ_1.	τ_2.	τ_3.	τ_4.	τ_5.	τ_6.	τ_7.
1	....	15.930	5.464	2.501	....	....	....	....
1′	$1 \times -\frac{5.464}{15.930}$ *i. e.*, 0.34300	[−5.464]	−1.874	−0.858	....	....	....	....
2	....	[+5.464]	+30.806	+3.303	+7.951	+1.685	....	....
I	1′ + 2	....	+28.932	−0.555	+7.951	+1.685	....	....
1″	$1 \times -\frac{2.501}{15.930}$ *i. e.*, 0.15700	[−2.501]	[−0.858]	−0.393	....	....	....	....
I′	$\mathrm{I} \times +\frac{0.555}{28.932}$ *i. e.*, 0.01918	....	[+0.555]	−0.011	+0.153	+0.032	....	....
3	....	[+2.501]	[+0.303]	+10.610	....	+2.501	....	....
II	1″ + I′ + 3	....	....	+10.206	+0.153	+2.533	....	....
I″	$\mathrm{I} \times -\frac{7.951}{28.932}$ *i. e.*, 0.27482	....	[−7.951]	[+0.153]	−2.185	−0.463	....	....
II′	$\mathrm{II} \times -\frac{0.153}{10.206}$ *i. e.*, 0.01499	....	....	[−0.153]	−0.002	−0.038	....	....
4	....	....	[+7.951]	....	+32.184	+0.190	+7.951	....
III	I′ + II′ + 4	....	....	....	+29.997	+0.311	+7.951	....
I‴	$\mathrm{I} \times -\frac{1.685}{28.932}$ *i. e.*, 0.05824	....	[−1.685]	[+0.032]	[−0.463]	−0.098	....	....
II″	$\mathrm{II} \times -\frac{2.533}{10.206}$ *i. e.*, 0.24819	....	....	[−2.533]	[−0.038]	−0.629	....	....
III′	$\mathrm{III} \times +\frac{0.311}{29.997}$ *i. e.*, 0.01037	....	....	....	[+0.311]	−0.003	+0.082	....
5	....	....	[+1.685]	[+2.501]	[+0.190]	+19.528	+1.800	+3.588
IV	I‴ + II″ + III′ + 5	....	....	....	....	+18.798	+1.882	+3.588
III″	$\mathrm{III} \times -\frac{7.951}{29.997}$ *i. e.*, 0.26506	....	....	....	[−7.951]	[+0.082]	−2.107	....
IV′	$\mathrm{IV} \times -\frac{1.882}{18.798}$ *i. e.*, 0.10012	....	....	....	....	[−1.882]	−0.188	−0.359
6	....	....	....	....	[+7.951]	[+1.800]	+39.610	+0.303
V	III″ + IV′ + 6	....	....	....	....	....	+37.315	−0.056
IV″	$\mathrm{IV} \times -\frac{3.588}{18.798}$ *i. e.*, 0.19087	....	....	....	....	[−3.588]	[−0.359]	−0.685
V′	$\mathrm{V} \times +\frac{0.056}{37.315}$ *i. e.*, 0.00150	....	....	....	....	....	[+0.056]	+0.000
7	....	....	....	....	....	[+3.588]	[+0.303]	+14.958
VI	IV″ + V′ + 7	....	....	....	....	....	....	+14.273
V″	$\mathrm{V} \times -\frac{1.800}{37.315}$ *i. e.*, 0.04824	....	....	....	....	....	[−1.800]	[+0.003]
VI′	$\mathrm{VI} \times -\frac{3.591}{14.273}$ *i. e.*, 0.25159	....	....	....	....	....	....	[−3.591]
8	....	....	....	....	....	....	[+1.800]	[+3.588]
VII	V″ + VI′ + 8	....	....	....	....	....	....	....
V‴	$\mathrm{V} \times -\frac{7.951}{37.315}$ *i. e.*, 0.21308	....	....	....	....	....	[−7.951]	[+0.012]
VI″	$\mathrm{VI} \times -\frac{0.012}{14.273}$ *i. e.*, 0.00084	....	....	....	....	....	....	[−0.012]
VII′	$\mathrm{VII} \times +\frac{0.197}{18.538}$ *i. e.*, 0.01063	....	....	....	....	....	....	....
9	....	....	....	....	....	....	[+7.951]	....
VIII	V‴ + VI″ + VII′ + 9	....	....	....	....	....	....	....
VII″	$\mathrm{VII} \times -\frac{2.501}{18.538}$ *i. e.*, 0.13491	....	....	....	....	....	....	....

EQUATIONS FOR INFLUENCE LINES.

τ_9.	τ_8.	τ_{11}.	τ_{10}.	τ_{12}.	1 000 LB. AT JOINT (7).		1 500 LB. AT JOINT (9).		3 000 LB. AT JOINT (11).	
					Absolute term.	Check.	Absolute term.	Check.	Absolute term.	Check.
....	....	....	....	....	+2.796	+26.691	....	....	....	....
....	....	....	....	....	−0.959	−9.155	....	....	....	....
....	....	....	....	....	−0.258	+45.951	....	....	....	....
....	....	....	....	....	−1.217	+36.796	....	....	....	....
....	....	....	....	....	−0.439	−4.190	....	....	....	....
....	....	....	....	....	−0.023	+0.706	....	....	....	....
....	....	....	....	....	−2.550	+13.365	....	....	....	....
....	....	....	....	....	−3.012	+9.881	....	....	....	....
....	....	....	....	....	+0.334	−10.112	+0.334	−10.112	....	....
....	....	....	....	....	+0.045	−0.148	+0.045	−0.148	....	....
....	....	....	....	....	+99.768	+140.093	+82.872	+131.148	....	....
....	....	....	....	....	+100.147	+129.833	+83.251	+120.888	....	....
....	....	....	....	....	+0.071	−2.143	+0.071	−2.143	....	....
....	....	....	....	....	+0.748	−2.452	+0.748	−2.452	....	....
....	....	....	....	....	+1.038	+1.346	+0.863	+1.253	....	....
....	....	....	....	....	+19.082	+42.986	−61.088	−31.796	....	....
....	....	....	....	....	+20.939	+39.737	−59.406	−35.138	....	....
....	....	....	....	....	....	....	−22.067	−32.043	−22.067	−32.043
....	....	....	....	....	....	....	+5.947	+3.518	+5.947	+3.518
+1.800	+7.951	....	....	....	....	....	+190.825	+250.240	+84.242	+143.657
+1.800	+7.951	....	....	....	....	....	+174.705	+221.715	+68.122	+115.132
....	....	....	....	....	....	....	+11.339	+6.707	+11.339	+6.707
+0.003	+0.012	....	....	....	....	....	+0.262	+0.333	+0.102	+0.173
+3.588	....	....	....	....	....	....	−44.304	−21.867	−44.304	−21.867
+3.591	+0.012	....	....	....	....	....	−32.703	−14.827	−32.863	−14.987
−0.087	−0.384	....	....	....	....	....	−8.427	−10.695	−3.286	−5.554
−0.903	−0.003	....	....	....	....	....	+8.228	+3.730	+8.268	+3.771
+19.528	+0.190	+2.501	+1.685	....	....	....	−162.576	−133.284	+22.760	+52.052
+18.538	−0.197	+2.501	+1.685	....	....	....	−162.775	−140.249	+27.742	+50.269
[−0.384]	−1.694	....	....	....	....	....	−37.226	−47.243	−14.515	−24.532
[−0.003]	−0.000	....	....	....	....	....	+0.027	+0.012	+0.028	+0.013
[+0.197]	−0.012	+0.027	+0.018	....	....	....	−1.730	−1.490	+0.295	+0.534
[+0.190]	+32.184	....	+7.951	....	....	....	+290.050	+338.326	+261.939	+310.215
....	+30.488	+0.027	+7.969	....	....	....	+251.121	+289.605	+247.747	+286.230
[−2.501]	[+0.027]	−0.337	−0.227	....	....	....	+21.960	+18.921	−3.743	−6.782

TABLE 25.—

Number of equation.	Remarks.	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7
VIII′	VIII $\times -\frac{0.027}{30.488}$ *i. e.*, 0.00089	....	....	....	....	....	....	....
10	....	....	....	....	....	....	....	....
IX	VII″ + VIII′ + 10	....	....	....	....	....	....	....
VII‴	VII $\times -\frac{1.685}{18.538}$ *i. e.*, 0.09089	....	....	....	....	....	....	....
VIII″	VIII $\times -\frac{7.969}{30.488}$ *i. e.*, 0.26138	....	....	....	....	....	....	....
IX′	IX $\times -\frac{0.069}{10.273}$ *i. e.*, 0.00672	....	....	....	....	....	....	....
11	....	....	....	....	....	....	....	....
X	VII‴ + VIII″ + IX′ + 11	....	....	....	....	....	....	....
IX″	IX $\times -\frac{2.501}{10.273}$ *i. e.*, 0.24345	....	....	....	....	....	....	....
X′	X $\times -\frac{5.447}{28.570}$ *i. e.*, 0.19065	....	....	....	....	....	....	....
12	....	....	....	....	....	....	....	....
XI	IX″ + X′ + 12	....	....	....	....	....	....	....
	1 000 lb. at Joint (7)..............	−0.6297	+1.0395	+0.6218	−3.3501	−1.1139	....	....
	1 500 lb. " Joint (9)..............	−0.2160	+0.3578	−0.5240	−1.9095	+3.4158	−3.1328	+0.3043
	3 000 lb. " Joint (11)..............	−0.3291	+0.5889	−0.3084	−2.5595	+2.5865	−0.7130	+3.3796

Table of τ-Values and Bending Moments.—A table similar to Table 6 can now be drawn up for the three cases, after which the $\Sigma\ M = 0$ test must be fulfilled for every joint.

Table of Secondary Stresses.—The bending moments found for Cases II and III are for 1 500 lb. and 3 000 lb., respectively. To obtain influence lines, the two sets are divided by 1.5 and 3, respectively. The resulting quantities are found in Table 26. The secondary stresses are found by multiplying the bending moments by $\frac{c}{I}$, in which c is the extreme fiber distance, and I the moment of inertia of the member under consideration.

Influence Lines.—Figs. 58 to 68 contain the influence lines for the primary stresses and those secondary stresses that are due to the rigidity of the joints in the plane of the truss.

Consider the end chord, 1-2, Fig. 58. Compressive primary stresses are drawn downward from a base line equal in length to the span of the bridge, while tensile primary stresses are drawn upward. The secondary stresses are drawn in just the reverse way, that is, compressive upward and tensile downward. To determine end effects, plot a set of lines for each end. The primary stress influence line, therefore, is drawn twice. Turning to Table 26, note that the primary stress in Member 1-2, is — 12.99 lb. per sq. in. for a load of 1 000 lb. at Joint (7), — 8.66 lb. per sq. in. for a load of 1 000 lb. at Joint (9), and — 4.33 lb. per sq. in. for a load of 1 000 lb. at Joint (11).

(*Continued*).

					1 000 lb. at Joint (7).		1 500 lb. at Joint (9).		3 000 lb. at Joint (11).	
τ_9	τ_8	τ_{11}	τ_{10}	τ_{12}	Absolute term.	Check.	Absolute term.	Check.	Absolute term.	Check.
....	[−0.027]	−0.000	−0.007	....	....	....	−0.222	−0.256	−0.219	−0.253
[+2.501]	....	+10.610	+0.303	+2.501	....	....	−48.694	−32.779	−291.127	−275.212
....	....	+10.273	+0.069	+2.501	....	....	−26.956	−14.114	−295.089	−282.247
[−1.685]	[+0.018]	[−0.227]	−0.153	....	....	....	+14.795	+12.748	−2.522	−4.569
....	[−7.969]	[−0.007]	−2.083	....	....	......	−65.638	−75.697	−64.756	−74.815
....	....	[−0.069]	−0.000	−0.017	....	....	+0.181	+0.095	+1.982	+1.896
[+1.685]	[+7.951]	[+0.303]	+30.806	+5.464	....	.....	−19.478	+26.731	+527.152	+573.361
....	....	....	+28.570	+5.447	....	....	−70.140	−36.123	+461.856	+495.873
....	....	[−2.501]	[−0.017]	−0.609	....	....	+6.563	+3.436	+71.841	+68.714
....	....	....	[−5.447]	+15.930	....	....	+13.372	+6.887	−88.055	−94.540
....	....	[+2.501]	[+5.464]	+15.930	....	....	+31.974	+55.869	+103.414	+127.309
....	....	....	....	+14.283	....	....	+51.909	+66.192	+87.200	+101.483
....	....	....	....	....	....	....	....	....	....	....
+7.9277	−9.0626	+3.4876	+3.1479	−3.6343	....	....	....	....	....	....
−4.2673	−4.2317	+30.3118	−15.0018	−6.1052	....	....	....	....	....	....

Now, when 1 000 lb. act at Joint (5), the effect on Member 1-2 is exactly the same as the effect of 1 000 lb. at Joint (9) on Member 10-12. From Table 26, this is — 17.32 lb. per sq. in. Finally, when 1 000 lb. act at Joint (3), the effect on Member 1-2 is exactly the same as the effect of 1 000 lb. at Joint (11) on Member 10-12, or — 21.65 lb. per sq. in.

The secondary stresses in Member 1-2 for End (1), are — 0.46, — 0.27 and — 0.13 lb. per sq. in., respectively, for the outside fiber stress, when the 1 000-lb. load is at Joints (7), (9), and (11), respectively, and + 0.47, + 0.28, and + 0.14 lb., respectively, for the inside fibers.

Again, 1 000 lb. acting at Joint (5) has exactly the same effect on End (1) as 1 000 lb. acting at Joint (9) has on End (12) of Member 12-10. It should be remembered that all the tabulations are for clockwise rotations around the joints, hence, the first set of fibers of Member 1-2, at End (1) corresponds to the last set of fibers of Member 12-10, at End (12). Therefore, take — 0.608 lb. per sq. in., found in Row 12-10, as the secondary stress in the outside fibers of End (1), when the 1 000 lb. act at Joint (5). The remainder follows in exactly the same way.

The reason for the reverse manner of representing the two kinds of stress is now evident; a glance at the four numbers that lie between the primary and secondary stress lines shows that the very maximum may happen to Member 1-2, when the bridge is fully loaded, and that it is the outside fibers of End (1), that are taxed more heavily than any of the three remaining sets.

TABLE 26.—Primary and Secondary Stresses for Influence Lines of Truss Members.

Joint.	Member.	$\frac{C}{I}$, in inch units.	1 000 lb. at Joint (7). Primary stress, in pounds per square inch.	1 000 lb. at Joint (7). Bending moment, in inch-pound units.	1 000 lb. at Joint (7). Secondary stress, in pounds per square inch.	1 000 lb. at Joint (9). Primary stress, in pounds per square inch.	1 000 lb. at Joint (9). Bending moment, in inch-pound units.	1 000 lb. at Joint (9). Secondary stress, in pounds per square inch.	1 000 lb. at Joint (11). Primary stress, in pounds per square inch.	1 000 lb. at Joint (11). Bending moment, in inch-pound units.	1 000 lb. at Joint (11). Secondary stress, in pounds per square inch.
1	1-2	0.003592 0.003634	− 12.99	− 129.27	− 0.464 + 0.470	− 8.66	− 75.55	− 0.271 + 0.275	− 4.33	− 37.59	− 0.135 + 0.137
	1-3	0.009997	+ 15.23	+ 129.27	± 1.292	+ 10.15	+ 75.57	± 0.755	+ 5.08	+ 37.62	± 0.376
2	2-4	0.003911 0.003999	− 18.61	+ 339.65	+ 1.328 − 1.358	− 12.41	+ 234.61	+ 0.918 − 0.938	− 6.20	+ 107.34	+ 0.420 − 0.429
	2-5	0.009312	+ 21.83	− 241.15	∓ 2.246	+ 14.55	− 89.62	∓ 0.835	+ 7.28	− 48.93	∓ 0.456
	2-3	0.046784	....	− 46.44	∓ 2.173	....	− 41.10	∓ 1.923	....	− 19.18	∓ 0.897
	2-1	0.003634 0.003592	− 12.99	− 52.09	− 0.189 + 0.187	− 8.66	− 103.89	− 0.378 + 0.373	− 4.33	− 39.23	− 0.143 + 0.141
3	3-1	0.009997	+ 15.23	+ 111.29	± 1.113	+ 10.15	+ 11.57	± 0.116	+ 5.08	+ 11.10	± 0.111
	3-2	0.046784	....	− 52 90	∓ 2.475	....	− 47.28	± 2.212	....	− 22.30	∓ 1.043
	3-5	0.009997	+ 15.23	− 58.34	∓ 0.583	+ 10.15	+ 35.75	± 0.357	+ 5.08	+ 11.25	± 0.112
4	4-6	0.003911 0.003999	− 18.61	− 111.50	− 0.436 + 0.446	− 12.41	− 326.41	− 1.277 + 1.305	− 6.20	− 103.84	− 0.406 + 0.415
	4-5	0.073492	....	− 71.84	∓ 5.279	....	− 29.12	∓ 2.140	....	− 17.26	∓ 1.268
	4-2	0.003999 0.003911	− 18.61	+ 183.40	+ 0.733 − 0.717	− 12.41	+ 355.36	+ 1.421 − 1.390	− 6.20	+ 121.01	+ 0.484 − 0.473
5	5-3	0.009997	+ 15.23	− 112.74	∓ 1.127	+ 10.15	+ 188.74	± 1.887	+ 5.08	+ 70.33	± 0.703
	5-2	0.009312	+ 21.83	− 313.72	∓ 2.921	+ 14.55	− 20.90	∓ 0.195	+ 7.28	− 26.49	∓ 0.247
	5-4	0.073492	....	− 76.28	∓ 5.606	....	− 24.51	∓ 1.801	....	− 15.05	∓ 1.106
	5-6	0.008717	− 19.85	− 375.33	∓ 3.272	− 13.23	− 158.97	∓ 1.386	− 6.62	− 70.34	∓ 0.613
	5-7	0.006968	+ 24.02	− 878.07	± 6.118	+ 16.01	+ 15.41	± 0.107	+ 8.01	+ 41.62	± 0.290
6	6-8	0.003911 0.003999	− 18.61	−1 375.189	− 5.378 + 5.499	− 24.81	+ 389.49	+ 1.523 − 1.558	− 12.41	− 3.18	− 0.012 + 0.013
	6-9	0.008717	− 19.85	− 3.39	∓ 0.030	+ 13.23	− 137.31	∓ 1.197	+ 6.62	− 156.91	+ 1.368
	6-7	0.046784	+ 41.07	....	....	....	− 65.49	∓ 3.064	....	− 16.76	∓ 0.784
	6-5	0.008717	− 19.85	+ 3.39	± 0.030	− 13.23	− 141.41	∓ 1.233	− 6.62	− 22.57	∓ 0.197
	6-4	0.003999 0.003911	− 18.61	+1 375.19	+ 5.499 − 5.378	− 12.41	− 45.37	− 0.181 + 0.177	− 6.20	+ 199.41	+ 0.797 − 0.780

TABLE 26—(*Continued*).

Joint.	Member.	$\frac{C}{I}$, in inch units.	1 000 lb. at Joint (7). Primary stress, in pounds per square inch.	1 000 lb. at Joint (7). Bending moment, in inch-pound units.	1 000 lb. at Joint (7). Secondary stress, in pounds per square inch.	1 000 lb. at Joint (9). Primary stress, in pounds per square inch.	1 000 lb. at Joint (9). Bending moment, in inch-pound units.	1 000 lb. at Joint (9). Secondary stress, in pounds per square inch.	1 000 lb. at Joint (11). Primary stress, in pounds per square inch.	1 000 lb. at Joint (11). Bending moment, in inch-pound units.	1 000 lb. at Joint (11). Secondary stress, in pounds per square inch.
7	7-5	0.006968	+24.02	+1 632.98	±11.379	+16.01	+29.52	±0.206	+8.01	+142.08	±0.990
	7-6	0.046784	+41.07	....	....	....	−67.26	∓3.147	....	−16.31	∓0.763
	7-9	0.006968	+24.02	−1 632.98	∓11.379	+16.01	+37.70	±0.263	+8.01	−125.70	∓0.876
8	8-10	0.003911	−18.61	−183.40	−0.717	−24.81	−1 792.26	−7.010	−12.41	−78.40	−0.307
		0.003999			+0.733			+7.167			+0.314
	8-9	0.073492	....	+71.84	±5.279	....	+16.76	±1.232	....	−28.79	∓2.116
	8-6	0.003999	−18.61	+111.50	+0.446	−24.81	+1 775.41	+7.100	−12.41	+107.02	+0.428
		0.003911			−0.436			−6.944			−0.419
9	9-7	0.006968	+24.02	−878.07	∓6.118	+16.01	+587.76	±4.096	+8.01	−215.95	∓1.505
	9-6	0.008717	−19.85	+375.33	±3.272	+13.23	+128.16	±1.117	+6.62	−199.56	∓1.740
	9-8	0.073492	....	+76.28	±5.606	....	+11.67	±0.858	....	−35.92	∓2.640
	9-10	0.009312	+21.83	+313.72	±2.921	+29.11	−105.25	∓0.980	−7.28	−234.39	∓2.183
	9-11	0.009997	+15.23	+112.74	±1.127	+20.31	−622.31	∓6.221	+25.38	+685.87	±6.857
10	10-12	0.003592	−12.99	+52.09	+0.187	−17.32	+428.67	+1.540	−21.65	−465.46	−1.672
		0.003634			−0.189			−1.558			+1.691
	10-11	0.046784	....	+46.44	±2.173	....	+102.49	±4.795	+41.07	−5.69	∓0.266
	10-9	0.009312	+21.83	+241.15	±2.246	+29.11	+170.97	±1.592	−7.28	−45.11	∓0.420
	10-8	0.003999	−18.61	−339.65	−1.358	−24.81	−702.13	−2.808	−12.41	+516.18	+2.064
		0.003911			+1.328			+2.746			−2.019
11	11-9	0.009997	+15.23	+58.34	±0.583	+20.31	−136.69	∓1.836	+25.38	+1 101.04	±11.007
	11-10	0.046784	....	+52.90	±2.475	....	+111.65	±5.223	+41.07	+10.57	±0.495
	11-12	0.009997	+15.23	−111.29	∓1.113	+20.31	+25.15	±0.251	+25.38	−1 111.63	∓11.113
12	12-11	0.009997	+15.23	−129.27	∓1.292	+20.31	−169.20	∓1.691	+25.38	−708.50	∓7.083
	12-10	0.003634	−12.99	+129.27	+0.470	−17.32	+169.31	+0.615	−21.65	+708.50	+2.575
		0.003592			−0.464			−0.608			−2.545

Fig. 58.

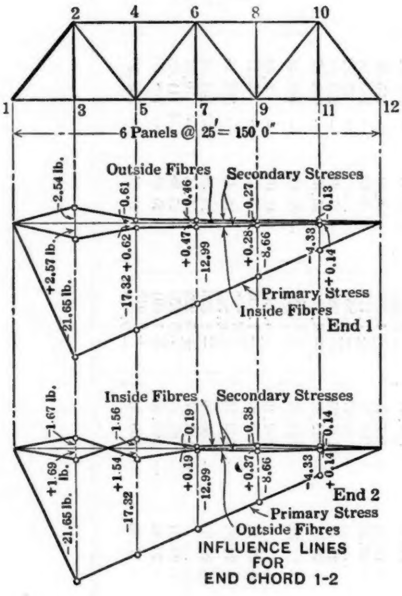
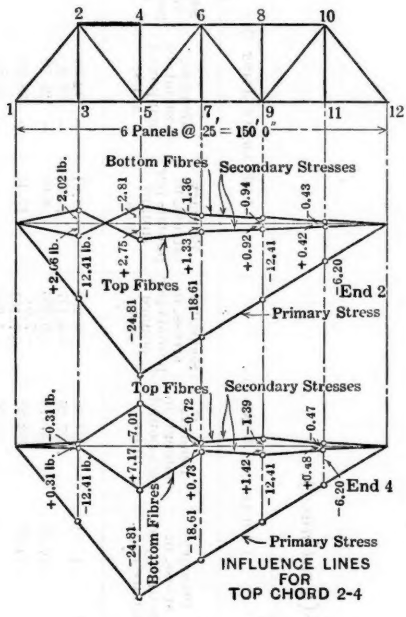


Fig. 59.



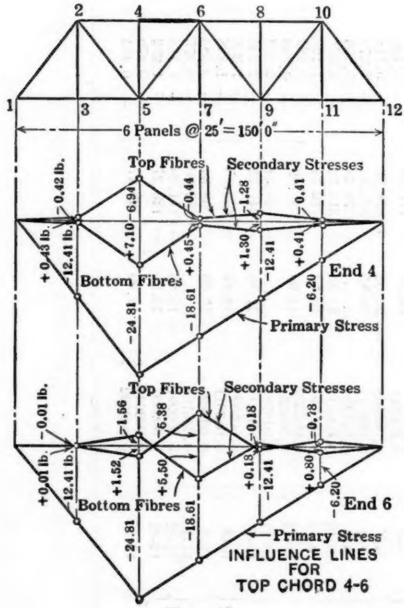


Fig. 60.

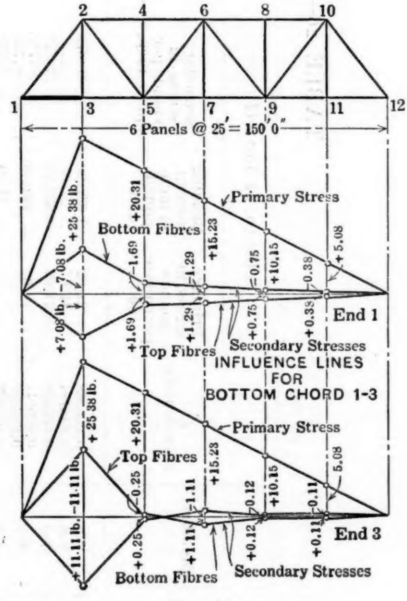


Fig. 61.

FIG. 62.

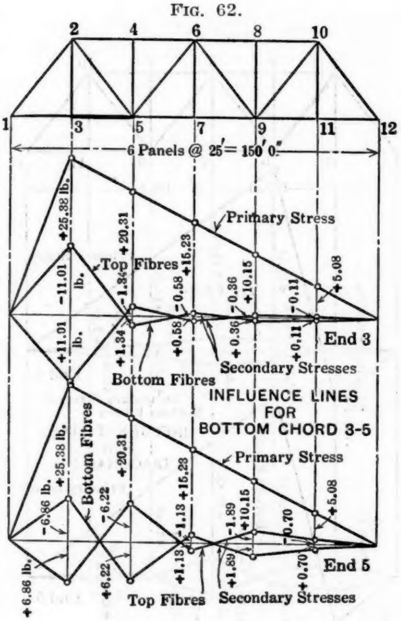
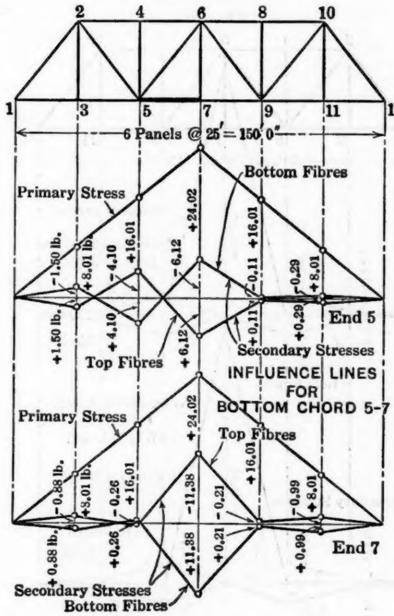


FIG. 63.



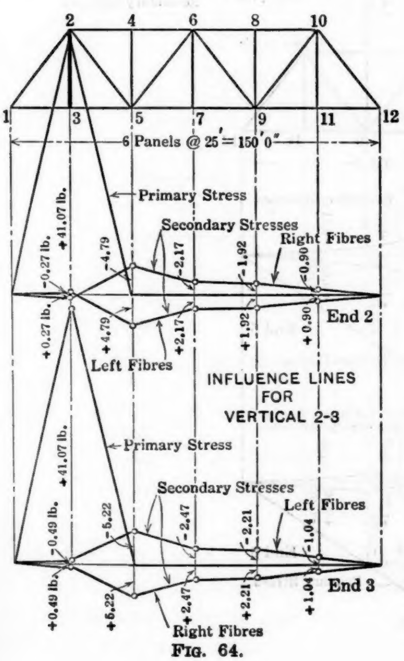


FIG. 64.

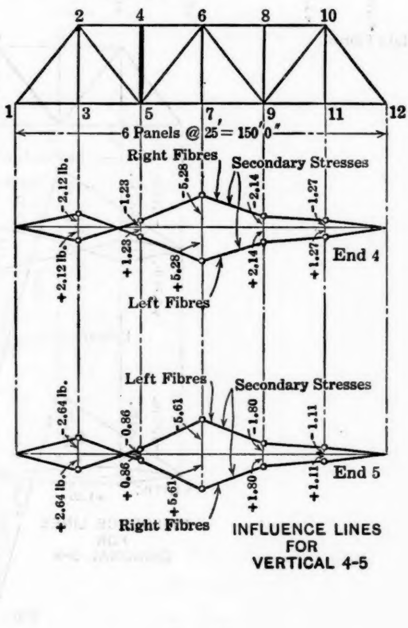


FIG. 65.

FIG. 66.

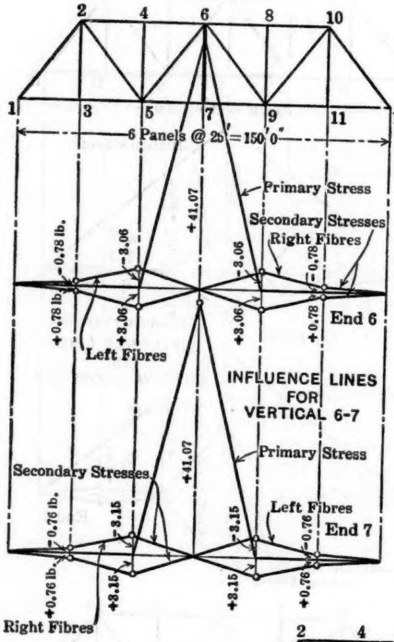
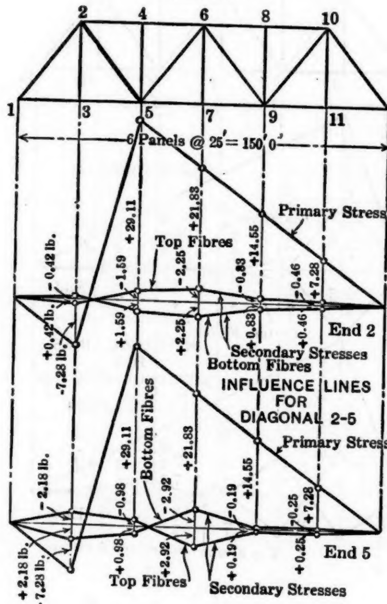


FIG. 67.



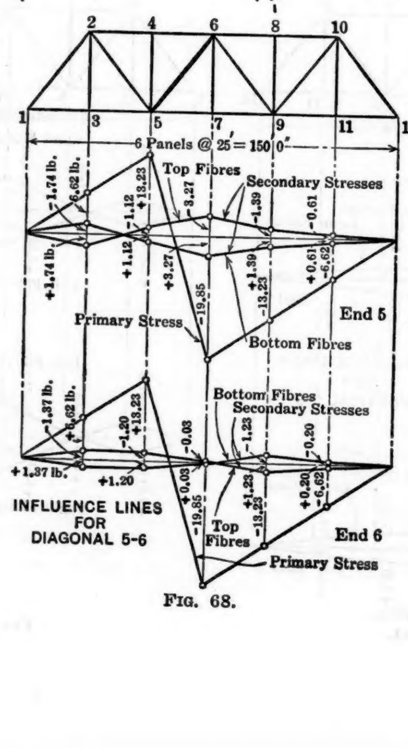


FIG. 68.

3.—Influence Lines for the Top Lateral System

The top lateral system forms a plane framework that has secondary stresses induced under either dead load or a combination of dead and live loads. The frame is considered complete from Joint (2) to Joint (10) of the bridge. Theoretically, the end portals contribute to the secondary stresses; their effect on the top lateral system, however, differs little from that found by reckoning the web-plates of the end portals as the closing members of the frame.

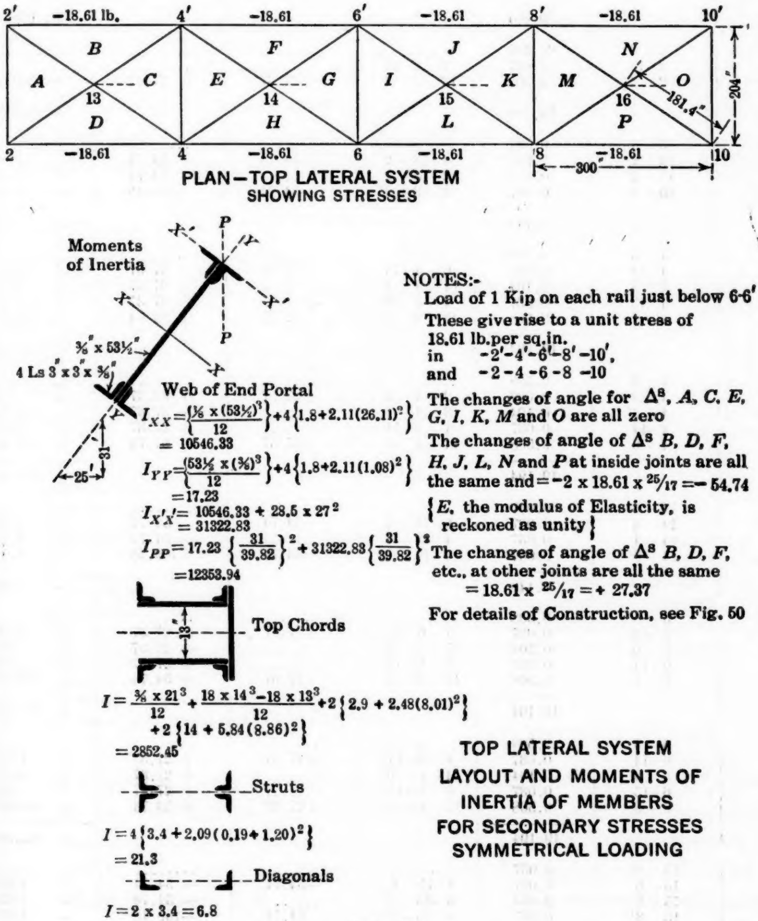


Fig. 69.

When there are two 1 000-lb. loads directly below Member 6-6′, Fig. 69, and symmetrical about the longitudinal center line, the unit stresses induced in the top chords are uniform and equal to —18.61 lb. per sq. in. Fig. 69 and Tables 27 to 29 represent the solution for this case. The determination of the new set of moments of inertia, characteristic of a horizontal frame, is to be noted.

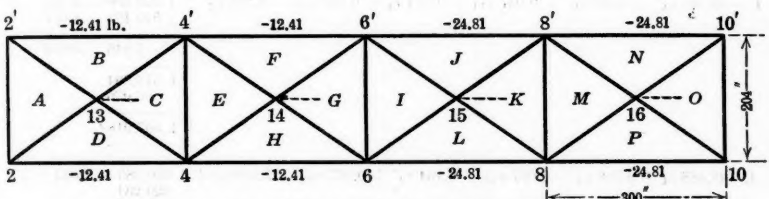
TABLE 27.—SECONDARY STRESSES IN TOP LATERAL SYSTEM DUE TO DEAD AND SYMMETRICAL LIVE LOADS. ELEMENTS OF MEMBERS.

Joint.	Member.	$K = \frac{I}{l}$	Angle.	δ Angle.	Σ δ Angle.	K Σ δ Angle.
2′	2′-4′	9.508	....	....	....	....
	2′-13	0.037	4′-2′-13	+27.37	+ 27.37	+ 1.013
	2′-2	60.559	13-2′-2	....	+ 27.37	+1 657.500
		70.104				+1 658.513
2	2-2′	60.559	....	....	....	....
	2-13	0.037	2′-2-13	....	....	....
	2-4	9.508	13-2-4	+27.37	+ 27.37	+ 260.234
		70.104				+ 260.234
13	13-4	0.037	....	....	....	....
	13-2	0.037	4-13-2	−54.74	− 54.74	− 2.025
	13-2′	0.037	2-13-2′	....	− 54.74	− 2.025
	13-4′	0.037	2′-13-4′	−54.74	−109.48	− 4.051
		0.148				− 8.102
4′	4′-6′	9.508	....	....	....	....
	4′-14	0.037	6′-4′-14	+27.37	+ 27.37	+ 1.013
	4′-4	0.104	14′-4′-4	....	+ 27.37	+ 2.846
	4′-13	0.037	4-4′-13	....	+ 27.37	+ 1.013
	4′-2′	9.508	13-4′-2′	+27.37	+ 54.74	+ 520.468
		19.194				+ 525.340
4	4-2	9.508	....	....	....	....
	4-13	0.037	2-4-13	+27.37	+ 27.37	+ 1.013
	4-4′	0.104	13-4-4′	....	+ 27.37	+ 2.846
	4-14	0.037	4′-4-14	....	+ 27.37	+ 1.013
	4-6	9.508	14-4-6	+27.37	+ 54.74	+ 520.468
		19.194				+ 525.340
14	14-6	0.037	....	....	....	....
	14-4	0.037	6-14-4	−54.74	− 54.74	− 2.025
	14-4′	0.037	4-14-4′	....	− 54.74	− 2.025
	14-6′	0.037	4′-14-6′	−54.74	−109.48	− 4.051
		0.148				− 8.102
6′	6′-8′	9.508	....	....	....	....
	6′-15	0.037	8′-6′-15	+27.37	+ 27.37	+ 1.013
	6′-6	0.104	15-6′-6	....	+ 27.37	+ 2.846
	6′-14	0.037	6-6′-14	....	+ 27.37	+ 1.013
	6′-4′	9.508	14-6′-4′	+27.37	+ 54.74	+ 520.468
		19.194				+ 525.340
6	6-4	9.508	....	....	....	....
	6-14	0.037	4-6-14	+27.37	+ 27.37	+ 1.013
	6-6′	0.104	14-6-6′	....	+ 27.37	+ 2.846
	6-15	0.037	6′-6-15	....	+ 27.37	+ 1.013
	6-8	9.508	15-6-8	+27.37	+ 54.74	+ 520.468
		19.194				+ 525.340
15	15-8	0.037	....	....	....	....
	15-6	0.037	8-15-6	−54.74	− 54.74	− 2.025
	15-6′	0.037	6-15-6′	....	− 54.74	− 2.025
	15-8′	0.037	6′-15-8′	−54.74	−109.48	− 4.051
		0.148				− 8.102
8′	8′-10′	9.508	....	....	....	....
	8′-16	0.037	10′-8′-16	+27.37	+ 27.37	+ 1.013
	8′-8	0.104	16-8′-8	....	+ 27.37	+ 2.846
	8′-15	0.037	8-8′-15	....	+ 27.37	+ 1.013
	8′-6′	9.508	15-8′-6′	+27.37	+ 54.74	+ 520.468
		19.194				+ 525.340

TABLE 27.—(*Continued*).

Joint.	Member.	$K = \frac{I}{l}$	Angle.	δ Angle.	$\Sigma\,\delta$ Angle.	$K\,\Sigma\,\delta$ Angle.
8	8 - 6	9.508	....	....	....	....
	8 -15	0.037	6 - 8 -15	+27.37	+ 27.37	+ 1.013
	8 - 8′	0.104	15 - 8 - 8′	....	+ 27.37	+ 2.846
	8 -16	0.037	8′ - 8 -16	....	+ 27.37	+ 1.013
	8 -10	9.508	16 - 8 -10	+27.37	+ 54.74	+ 520.468
		19.194				+ 525.340

When there are two 1 000-lb. loads directly below Member 8-8′, Fig. 70, symmetrical about the longitudinal center line, the unit stresses in Members 2-4, 4-6, 2′-4′, and 4′-6′, are the same and equal to — 12.41 lb. per sq. in., while the unit stresses in Members 6-8, 8-10, 6′-8′, and 8′-10′, are the same and equal to — 24.81 lb. per sq. in. Fig. 70 and Tables 30 to 33 represent the solution for this case.



PLAN TOP LATERAL SYSTEM SHOWING STRESSES

NOTES:

Load of 1 Kip on each rail just below 8-8′.

The changes of angle in all the Δs A, C, E, G, I, K, M and O are all zero.

The changes of angle of B_2', B_4', F_4', F_6', D_2, D_4, H_4 and H_6 are each + 18.25.

The changes of angle of J_6', J_8', N_8', N_{10}', L_6, L_8, P_8 and P_{10} are each + 36.49.

The changes of angle of B_{13}, D_{13}, F_{14} and H_{14} are each - 36.50; those of J_{15}, L_{15}, N_{16}, P_{16} are each - 72.98.

For details of construction see Fig. 50.

For Moments of Inertia of Members see Fig. 69.

TOP LATERAL SYSTEM
LAYOUT FOR OBTAINING
SECONDARY STRESSES
UNSYMMETRICAL LOADING

FIG. 70.

When there are two 1 000-lb. loads directly below Member 10-10′, symmetrical about the longitudinal center line, the unit stresses in the top chords are exactly one-half those of the former case (the 1 000-lb. loads directly below Member 8-8′). The previous solution therefore determines the effect of loads directly below Members 2-2′ and 10-10′, as the bending moments and secondary stresses are just half what they are when the loads are directly below Member 4-4′ and Member 8-8′ respectively.

Therefore, only two sets of solutions are required, which are extremely easy, as may be noted. The ease with which Manderla's method of solution is used when there are interior joints is worthy of note.

In this analysis for the top lateral system, the influence lines for Members 2-4 and 4-6 are the only ones required for the chord members, as the loading

TABLE 28.—Secondary Stresses in Top Lateral System Due to Dead and Symmetrical Live Loads. Computation of T's.

$$2'.\ -140.208\,\tau_2' + 9.508\,\tau_4' + 0.037\,\tau_{13} + 60.559\,\tau_2 + 3\,317.025 - 2.025 = 0$$

520.468
3 837.493
2.025
+ 3 835.468

$$2.\ -140.208\,\tau_2 + 60.559\,\tau_2' + 0.037\,\tau_{13} + 9.508\,\tau_4 + 520.468 - 2.025 = 0$$

1 657.500
2 177.968
2.025
+ 2 175 943

$$13.\ -0.296\,\tau_{13} + 0.037\,\tau_4 + 0.037\,\tau_2 + 0.037\,\tau_2' + 0.037\,\tau_4' - 16.203 + 1.013 = 0$$

3.038   1.013
− 13.165   1.013
1.013
3.038

$$4'.\ -38.388\,\tau_4' + 9.508\,\tau_6' + 0.037\,\tau_{14} + 0.104\,\tau_4 + 0.037\,\tau_{13} + 9.508\,\tau_2' + 1\,050.680 - 2.025 = 0$$

520.468   4.051
2.846   6.076
1 573.994
6.076
+ 1 567.918

$$4.\ -38.388\,\tau_4 + 9.508\,\tau_2 + 0.037\,\tau_{13} + 0.104\,\tau_4' + 0.037\,\tau_{14} + 9.508\,\tau_6 + 1\,050.680 - 2.025 = 0$$

260.234
2.846
1 313.760
2.025
+ 1 311.735

$$14.\ -0.296\,\tau_{14} + 0.037\,\tau_6 + 0.037\,\tau_4 + 0.037\,\tau_4' + 0.037\,\tau_6' - 16.203 + 1.013 = 0$$

4.051   1.013
− 12.152   1.013
1.013
4.051

$$6'.\ -38.388\,\tau_6' + 9.508\,\tau_8' + 0.037\,\tau_{15} + 0.104\,\tau_6 + 0.037\,\tau_{14} + 9.508\,\tau_4' + 1\,050.680 - 2.025 = 0$$

520.468   4.051
2.846   6.076
+ 1 573.994
6.076
+ 1 567.918

$$6.\ -38.388\,\tau_6 + 9.508\,\tau_4 + 0.037\,\tau_{14} + 0.104\,\tau_6' + 0.037\,\tau_{15} + 9.508\,\tau_8 + 1\,050.680 - 2.025 = 0$$

520.468
2.846
+ 1 573.994
2.025
+ 1 571.969

TABLE 28.—(*Continued*).

From Symmetry:

$$
\begin{aligned}
\tau_8' &= \tau_{8'\cdot 10'} = -\tau_{4'2'} \\
&= -(\tau_4' + 54.74) \\
&= -\tau_4' - 54.74, \\
\tau_8 &= \tau_{86} = -\tau_{46} \\
&= -\tau_4 - 54.74, \\
\tau_{15} &= \tau_{15\cdot 8} = -\tau_{14\cdot 4} = -\tau_{14} + 54.74, \\
\tau_{24} &= -\tau_{2'4'} \\
&\text{Therefore } \tau_2' = -\tau_2 - 27.37, \\
\tau_{42} &= -\tau_{4'2'} \\
&\text{Therefore } \tau_4 = -\tau_4' - 54.74, \\
\tau_{6\cdot 4} &= -\tau_{6'4'} \\
&\text{Therefore } \tau_6 = -\tau_6' - 54.74.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
&2.\ -140.208\,\tau_2 + 60.559\,(-\tau_2 - 27.37) + 0.037\,\tau_{13} + 9.508\,\tau_4 + 2\,175.943 = 0 \\
&13.\ -0.296\,\tau_{13} + 0.037\,\tau_4 + 0.037\,\tau_2 + 0.037\,(-\tau_2 - 27.37) + 0.037\,(-\tau_4 - 54.74) - 13.165 = 0 \\
&4.\ -38.388\,\tau_4 + 9.508\,\tau_2 + 0.037\,\tau_{13} + 0.104\,(-\tau_4 - 54.74) + 0.037\,\tau_{14} + 9.508\,\tau_6 + 1\,311.735 = 0 \\
&14.\ -0.296\,\tau_{14} + 0.037\,\tau_6 + 0.037\,\tau_4 + 0.037\,(-\tau_4 - 54\ 74) + 0.037\,(-\tau_6 - 54.74) - 12.152 = 0 \\
&6.\ -38.388\,\tau_6 + 9.508\,\tau_4 + 0.037\,\tau_{14} + 0.104\,(-\tau_6 - 57.74) + 0.037\,(-\tau_{14} + 54.74) \\
&\qquad + 9.508\,(-\tau_4 - 54.74) + 1\,571.969 = 0
\end{aligned}
$$

that is.

$$
\begin{aligned}
&2.\ +79.649\,\tau_2 + 0.037\,\tau_{13} + 9.508\,\tau_4 + 518.443 = 0 \\
&13.\ +0.296\,\tau_{13} - 16.203 = 0 \\
&4.\ +38.284\,\tau_4 + 9.508\,\tau_2 + 0.037\,\tau_{13} + 0.037\,\tau_{14} + 9.508\,\tau_6 + 1\,306.042 = 0 \\
&14.\ +0.296\,\tau_{14} - 16.203 = 0 \\
&6.\ +38.284\,\tau_6 + 1\,047.834 = 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Therefore } \tau_{13} &= \tau_{14} = +54.74, \\
\tau_6 &= -27.37, \\
\text{and } +79.649\,\tau_2 + 9.508\,\tau_4 + 520.468 &= 0 \\
+9.508\,\tau_2 + 38.284\,\tau_4 + 1\,049.859 &= 0, \\
\text{Therefore } \tau_4 &= -26.588, \\
\tau_2 &= -\ \ 3.361.
\end{aligned}
$$

$$
\text{Therefore}\ \left\{
\begin{aligned}
\tau_2 &= -\ \ 3.361 \\
\tau_4 &= -26.588 \\
\tau_6 &= -27.870 \\
\tau_{13} &= +54.740 \\
\tau_{14} &= +54.740 \\
\tau_2' &= -24.009 \\
\tau_4' &= -28.152 \\
\tau_6' &= -27.370 \\
\tau_{15} &= \ \ \ldots\ldots \\
\tau_8 &= -28.152 \\
\tau_8' &= -26.588
\end{aligned}
\right.
$$

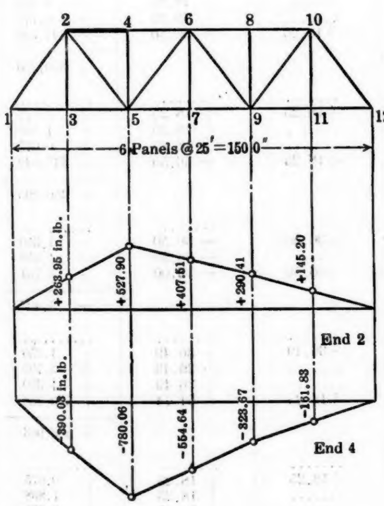
is symmetrical about the longitudinal center line. If necessary, the influence lines for the struts and diagonals could easily be drawn. The influence lines for Members 2-4 and 4-6 are given in Figs. 71 and 72. They have been purposely drawn for bending moments, and not for secondary stresses. From Fig. 69 it is seen that the extreme fiber distances of the cover-plate and the bottom angles of the chords are, respectively, 10.5 in. and 12 in. from the neutral plane under consideration. As the moment at the end of a member is numerically the same, no matter which extreme fiber distance is considered, an influence line for bending moment can be converted at once into the secondary stress influence line for a particular extreme fiber by suitably changing the scale. Hence, by multiplying the quantities for End (4) by $\pm \dfrac{10.5}{2\ 852.45}$, the secondary stress influence lines for the extreme fibers of the cover-plate are obtained, and if by $\pm \dfrac{12}{2\ 852.45}$, those for the extreme fibers of the bottom angles.

TABLE 29.—SECONDARY STRESSES IN TOP LATERAL, ETC. VALUES OF MOMENTS.

2'........	$\tau_{2'4'} = -24.009$ $+27.370$	$M_{2'4'} = 19.016(-48.018+26.588) = -407.51$
	$\tau_{2'.13} = +3.361$	$M_{2'.13} = 0.074(+6.724+0) = +0.50$
	$\tau_{2'2} = +3.361$	$M_{2'2} = 121.118(+6.724-3.361) = +407.08$
2.............	$\tau_{22'} = -3.361$	$M_{22'} = 121.118(-6.724+3.361) = -407.08$
	$\tau_{2.13} = -3.361$ $+27.370$	$M_{2.13} = 0.074(-6.724+0) = -0.50$
	$\tau_{24} = +24.009$	$M_{24} = 19.016(+48.018-26.588) = +407.51$
13.............	$\tau_{13.4} = +54.740$ -54.740	$M_{13.4} = 0.074(+109.48+0.782) = +8.16$
	$\tau_{13.2} =$	$M_{13.2} = 0.074(0-3.361) = -0.25$
	$\tau_{13.2'} =$ ...... -54.740	$M_{13.2'} = 0.074(0+3.361) = +0.25$
	$\tau_{13.4'} = -54.740$	$M_{13.4'} = 0.074(-109.48-0.782) = -8.16$
4'.............	$\tau_{4'6'} = -28.152$ $+27.370$	$M_{4'6'} = 19.016(-56.304+27.370) = -550.21$
	$\tau_{4'.14} = -0.782$	$M_{4'.14} = 0.074(-1.564+0) = -0.12$
	$\tau_{4'.4} = -0.782$	$M_{4'.4} = 0.208(-1.564+0.782) = -0.16$
	$\tau_{4'.13} = -0.782$ $+27.370$	$M_{4'.13} = 0.074(-1.564-54.740) = -4.17$
	$\tau_{4'.2'} = +26.588$	$M_{4'.2'} = 19.016(+53.176-24.009) = +554.64$
4.............	$\tau_{4.2} = -26.588$ $+27.370$	$M_{42} = -554.64$
	$\tau_{4.13} = +0.782$	$M_{4.13} = +4.17$
	$\tau_{44'} = +0.782$	$M_{44'} = +0.16$
	$\tau_{4.14} = +0.782$ $+27.370$	$M_{4.14} = +0.12$
	$\tau_{4.6} = +28.152$	$M_{4.6} = +550.21$
14.............	$\tau_{14.6} = +54.74$ -54.74	$M_{14.6} = 0.074(+109.48+0) = +8.10$
	$\tau_{14.4} =$	$M_{14.4} = 0.074(0+0.782) = +0.06$
	$\tau_{14.4'} =$ ...... -54.74	$M_{14.4'} = -0.06$
	$\tau_{14.6'} = -54.74$	$M_{14.6'} = -8.10$
6'.............	$\tau_{6'8'} = -27.37$ $+27.37$	$M_{6'8'} = 19.016(-54.74+28.152) = -505.60$
	$\tau_{6'.15} =$	$M_{6'.15} = 0.074(0+54.74) = +4.05$
	$\tau_{6'.6} =$	$M_{6'.6} = 0.208(0+0) =$ ....
	$\tau_{6'.14} =$ ...... $+27.37$	$M_{6'.14} = -4.05$
	$\tau_{6'.4} = +27.37$	$M_{6'4'} = +505.60$

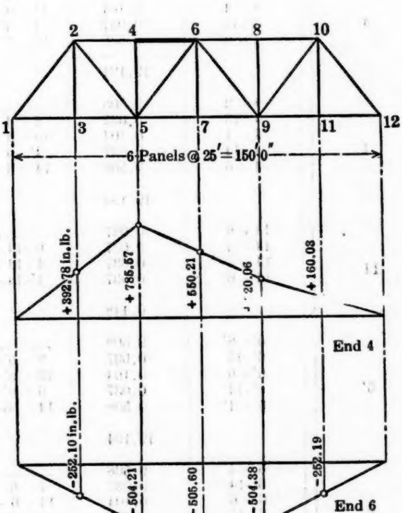
TABLE 29.—(*Continued*).

6	$\tau_{6.4}$ = − 27.37 + 27.37	M_{64} =	− 505.60
	$\tau_{6.14}$ = ...	$M_{6.14}$ =	+ 4.05
	$\tau_{66'}$ = ...	$M_{66'}$ =	....
	$\tau_{6.15}$ = ... + 27.37	$M_{6.15}$ =	− 4.05
	$\tau_{6.8}$ = + 27.37	$M_{6.8}$ =	+ 505.60



INFLUENCE LINES OF BENDING MOMENTS INDUCED IN 2-4 AS PART OF TOP LATERAL SYSTEM

FIG. 71.



INFLUENCE LINES OF BENDING MOMENTS INDUCED IN 4-6 AS PART OF TOP LATERAL SYSTEM

FIG. 72.

TABLE 30.—SECONDARY STRESSES IN TOP LATERAL SYSTEM DUE TO DEAD AND UNSYMMETRICAL LIVE LOADS. ELEMENTS OF MEMBERS.

Joint.	Member.	$K = \frac{I}{l}$	Angle.	δ Angle.	Σ δ Angle.	K Σ δ Angle.
2′	2′- 4′	9.508	....	....	....	....
	2′-13	0.037	4′- 2′-13	+18.25	+ 18.25	+ 0.675
	2′- 2	60.559	13 - 2′- 2	....	+ 18.25	+1105.202
		70.104				+1105.877
2	2 - 2′	60.559	....	....	....	....
	2 -13	0.037	2′- 2 -13	....	....	....
	2 - 4	9.508	13 - 2 - 4	+18.25	+ 18.25	+ 173.521
		70.104				+ 173.521
13	13 - 4	0.037	....	....	....	....
	13 - 2	0.037	4 -13 - 2	−36.50	− 36.50	− 1.350
	13 - 2′	0.037	2 -13 - 2′	....	− 36.50	− 1.350
	13 - 4′	0.037	2′-13 - 4′	−36.50	− 73.00	− 2.701
		0.148				− 5.402
4′	4′- 6′	9.508	....	....	....	....
	4′-14	0.037	6′- 4′-14	+18.25	+ 18.25	+ 0.675
	4′- 4	0.104	14 - 4′- 4	....	+ 18.25	+ 1.898
	4′-13	0.037	4 - 4′-13	....	+ 18.25	+ 0.675
	4′- 2′	9.508	13 - 4′- 2′	+18.25	+ 36.50	+ 347.042
		19.194				+ 350.290
4	4 - 2	9.508	....	....	....	....
	4 -13	0.037	2 - 4 -13	+18.25	+ 18.25	+ 0.675
	4 - 4′	0.104	13 - 4 - 4′	....	+ 18.25	+ 1.898
	4 -14	0.037	4′- 4 -14	....	+ 18.25	+ 0.675
	4 - 6	9.508	14 - 4 - 6	+18.25	+ 36.50	+ 347.042
		19.194				+ 350.290
14	14 - 6	0.037	....	....	....	....
	14 - 4	0.037	6 -14 - 4	−36.50	− 36.50	− 1.350
	14 - 4′	0.037	4 -14 - 4′	....	− 36.50	− 1.350
	14 - 6′	0.037	4′-14 - 6′	−36.50	− 73.00	− 2.701
		0.148				− 5.402
6′	6′- 8′	9.508	....	....	....	....
	6′-15	0.037	8′- 6′-15	+36.49	+ 36.49	+ 1.350
	6′- 6	0.104	15 - 6′- 6	....	+ 36.49	+ 3.795
	6′-14	0.037	6 - 6′-14	....	+ 36.49	+ 1.350
	6′- 4′	9.508	14 - ′6- 4′	+18.25	+ 54.74	+ 520.468
		19.194				+ 526.963
6	6 - 4	9.508	....	....	....	....
	6 -14	0.037	4 - 6 -14	+18.25	+ 18.25	+ 0.675
	6 - 6′	0.104	14 - 6 - 6′	....	+ 18.25	+ 1.898
	6 -15	0.037	6′- 6 -15	....	+ 18.25	+ 0.675
	6 - 8	9.508	15 - 6 - 8	+36.49	+ 54.74	+ 520.468
		19.194				+ 523.716
15	15 - 8	0.037	....	....	....	....
	15 - 6	0.037	8 -15 - 6	−72.98	− 72.98	− 2.700
	15 - 6′	0.037	6 -15 - 6′	....	− 72.98	− 2.700
	15 - 8′	0.037	6′-15 - 8′	−72.98	−145.96	− 5.401
		0.148				− 10.801

TABLE 30.—(*Continued.*)

Joint.	Member.	$K = \frac{I}{l}$	Angle.	δ Angle.	$\Sigma\ \delta$ Angle.	$K\ \Sigma\ \delta$ Angle.
8′	8′-10′	9.508	....	....	....	....
	8′-16	0.037	10′- 8′-16	+36.49	+ 36.49	+ 1.350
	8′- 8	0.104	16 - 8′- 8	....	+ 36.49	+ 3.795
	8′-15	0.037	8 - 8′-15	....	+ 36.49	+ 1.850
	8′- 6′	9.508	15 - 8′- 6′	+36.49	+ 72.98	+ 693.894
		19.194				+ 700.389
8	8 - 6	9.508	....	....	....	....
	8 -15	0.037	6 - 8 -15	+36.49	+ 36.49	+ 1.350
	8 - 8′	0.104	15 - 8 - 8′	....	+ 36.49	+ 3.795
	8 -16	0.037	8′- 8 -16	....	+ 36.49	+ 1.350
	8 -10	9.508	16 - 8 -10	+36.49	+ 72.98	+ 698.894
		19.194				+ 700.389
16	16 -10	0.037	....	....	....	....
	16 - 8	0.037	10 -16 - 8	−72.98	− 72.98	− 2.700
	16 - 8′	0.037	8 -16 - 8′	....	− 72.98	− 2.700
	16 -10′	0.037	8′-16 -10′	−72.98	−145.96	− 5.401
		0.148				− 10.801
10′	10′-10	60.559	....	....	....	....
	10′-16	0.037	10 -10′-16	....	....	....
	10′- 8′	9.508	16 -10′- 8′	+36.49	+ 36.49	+ 346.947
		70.104				+ 346.947
10	10 - 8	9.508	....	....	....	....
	10 -16	0.037	8 -10 -16	+36.49	+ 36.49	+ 1.350
	10 -10′	60.559	16 -10 -10′	....	+ 36.49	+2209.798
		70.104				+2211.148

Tables 29 and 33 permit the construction of the influence lines as follows: When the loads are below Member 6-6′, $M_{46} = +\,550.21$ and $M_{64} = -\,505.60$ in-lb. When the loads are below Member 8-8′, $M_{46} = +\,320.06$ and $M_{64} = -\,504.38$ in-lb. When the loads are below Member 10-10′, these last two quantities are halved, giving $+\,160.03$ and $-\,252.19$, respectively. When the loads are below Member 4-4′, the effects on End (4) and End (6) of Member 4-6 are the same as the effects on End (8) and End (6), respectively, of Member 8-6 when the loads are situated below Member 8-8′. Here, there is a slight variation, depending on the convention of signs. M_{86} is given as $-\,785.57$ in-lb., Table 33, that is, a bending moment that is acting in a clockwise direction around Joint (8); the corresponding one around Joint (4), therefore, must be counterclockwise and reckoned as $+\,785.57$ in-lb. The remaining influence lines can be easily deduced.

4.—Influence Lines for Cross-Frame 6-6′-7′-7

The effect of the slender angles that extend from the top lateral strut to the vertical posts is neglected; the secondary stresses due to them are extremely small. Therefore, the problem concerns a rectangular frame, as shown in Fig. 73. Consider the unsymmetrical case of a load on one rail only; not only will this determine the best method under the worst conditions, but it may easily be extended to take into account wind forces.

TABLE 31.—Secondary Stresses in Top Lateral System Due to Dead and Unsymmetrical Live Loads. Formation of Equations.

$$2'.\quad +140.208\,\tau_2' + 9.508\,\tau_4' + 0.037\,\tau_{13} + 60.559\,\tau_2 + 2\,211.754 - 1.350 = 0$$

347.042
2 558.796
1.350
+ 2 557.446

$$2.\quad +140.208\,\tau_2 + 60.559\,\tau_2' + 0.037\,\tau_{13} + 9.508\,\tau_4 + 347.042 - 1.350 = 0$$

1 105.202
1 452.244
1.350
+ 1 450.894

$$13.\quad +0.296\,\tau_{13} + 0.037\,\tau_4 + 0.037\,\tau_2 + 0.037\,\tau_2' + 0.037\,\tau_4' - 10.804 + 0.675 = 0$$

2.025	0.675
− 8.779	0.675
	2.025

$$4'.\quad +38.388\,\tau_4' + 9.508\,\tau_6' + 0.037\,\tau_{14} + 0.104\,\tau_4 + 0.037\,\tau_{13} + 9.508\,\tau_2' + 700.580 - 1.350 = 0$$

520.468	2.701
1.898	4.051
+ 1 222.946	
4.051	
+ 1 218.895	

$$4.\quad +38.388\,\tau_4 + 9.508\,\tau_6 + 0.037\,\tau_{14} + 0.104\,\tau_4' + 0.037\,\tau_{13} + 9.508\,\tau_2 + 700.580 - 1.350 = 0$$

173.521
1.898
875.999
1.350
+ 874.649

$$14.\quad +0.296\,\tau_{14} + 0.037\,\tau_6 + 0.037\,\tau_4 + 0.037\,\tau_4' + 0.037\,\tau_6' - 10.804 + 0.675 = 0$$

3.376	0.675
− 7.428	0.675
	1.350
	3 376

$$6'.\quad +38.388\,\tau_6' + 9.508\,\tau_8' + 0.037\,\tau_{15} + 0.104\,\tau_6 + 0.037\,\tau_{14} + 9.508\,\tau_4' + 1\,053.926 - 2.700 = 0$$

693.894	2.701
1.898	5.401
1 749.718	
5.401	
+ 1 744.317	

$$6.\quad +38.388\,\tau_6 + 9.508\,\tau_4 + 0.037\,\tau_{14} + 0.104\,\tau_6' + 0.037\,\tau_{15} + 9.508\,\tau_8 + 1\,047.432 - 2.700 = 0$$

347.042
3.795
+ 1 398.269
2.700
+ 1 395.569

TABLE 31.—(*Continued*).

15. $+0.296\,\tau_{15} + 0.037\,\tau_8 + 0.037\,\tau_6 + 0.037\,\tau_6' + 0.037\,\tau_8' - 21.602 + 0.675 = 0$

$$\begin{array}{rr} -21.602 & +0.675 \\ 4.725 & 1.350 \\ \hline -16.877 & 1.350 \\ & 1.350 \\ \hline & 4.725 \end{array}$$

8'. $+38.388\,\tau_8' + 9.508\,\tau_{10}' + 0.037\,\tau_{16} + 0.104\,\tau_8 + 0.037\,\tau_{15} + 9.508\,\tau_6' + 1\,400.778 - 2.700 = 0$

$$\begin{array}{rr} 1\,400.778 & -2.700 \\ 346.947 & 5.401 \\ & \hline 3.795 & 8.101 \\ \hline 1\,751.520 & \\ 8.101 & \\ \hline +1\,743.419 & \end{array}$$

8. $+38.388\,\tau_8 + 9.508\,\tau_6 + 0.037\,\tau_{15} + 0.104\,\tau_8' + 0.037\,\tau_{16} + 9.508\,\tau_{10} + 1\,400.778 - 2.700 = 0$

$$\begin{array}{r} 1\,400.778 \\ 520.468 \\ 3.795 \\ \hline 1\,925.041 \\ 2.700 \\ \hline +1\,922.341 \end{array}$$

16. $+0\,296\,\tau_{16} + 0.037\,\tau_{10} + 0.037\,\tau_8 + 0.037\,\tau_8' + 0.037\,\tau_{10}' - 21.602 + 1.350 = 0$

$$\begin{array}{rr} -21.602 & +1.350 \\ 4.050 & 1.350 \\ \hline -17.552 & 1.350 \\ & \hline & 4\,050 \end{array}$$

10'. $+140.208\,\tau_{10}' + 60.559\,\tau_{10} + 0.037\,\tau_{16} + 9.508\,\tau_8' + 693.894 - 5.401 = 0$

$$\begin{array}{r} 693.894 \\ 2\,209.798 \\ \hline 2\,903.692 \\ 5.401 \\ \hline +2\,898.291 \end{array}$$

10. $+140.208\,\tau_{10} + 60.559\,\tau_{10}' + 0.037\,\tau_{16} + 9.508\,\tau_8 + 4\,422.296 = 0$

$$\begin{array}{r} 4\,422.296 \\ 693.894 \\ \hline +5\,116.190 \end{array}$$

FROM SYMMETRY:

$$\left.\begin{array}{l} \tau_2' = -\tau_2 - 18.25 \\ \tau_4' = -\tau_4 - 36.50 \\ \tau_6' = -\tau_6 - 54.74 \\ \tau_8' = -\tau_8 - 72.98 \\ \tau_{10}' = -\tau_{10} - 36.49 \end{array}\right\}$$

Equation (2') becomes:

$$+140.208\,(-\tau_2 - 18.25) + 9.508\,(-\tau_4 - 36.50) + 0.037\,\tau_{13} + 60.559\,\tau_2 + 2\,557.446 = 0,$$

that is, $+79.649\,\tau_2 + 9.508\,\tau_4 - 0.037\,\tau_{13} + 348.392 = 0.$

Equation (2) becomes:

$$+79.649\,\tau_2 + 9.508\,\tau_4 + 0.037\,\tau_{13} + 345.692 = 0.$$

Combined:

$$+79.649\,\tau_2 + 9.508\,\tau_4 + 347.042 = 0.$$

Similarly, Equations (4') and (4) give:

$$+38.284\,\tau_4 + 9.508\,\tau_6 + 9.508\,\tau_2 + 873.554 = 0,$$

Equations (6') and (6) give:

$$+38.284\,\tau_6 + 9.508\,\tau_8 + 9.508\,\tau_4 + 1\,393.927 = 0,$$

Equations (8') and (8) give:

$$+38.284\,\tau_8 + 9.508\,\tau_{10} + 9.508\,\tau_6 + 1\,920.151 = 0,$$

Equations (10') and (10) give:

$$+79.649\,\tau_{10} + 9.508\,\tau_8 + 2\,909.092 = 0.$$

TABLE 32.—SECONDARY STRESSES IN TOP LATERAL SYSTEM DUE TO DEAD AND UNSYMMETRICAL LIVE LOADS. COMPUTATION OF T'S.

No. of Equation.	Remarks.	τ_2.	τ_4.	τ_6.	τ_8.	τ_{10}.	Absolute term.	Check
2	....	+79.649	+ 9.508	....	....	....	+ 347.042	+ 436.199
2′	$2 \times -\frac{9.508}{79.649}$ *i. e.*, — 0.119374	[— 9.508]	— 1.135	....	....	....	— 41.428	— 52.071
4	....	[+ 9.508]	+38.284	+ 9.508	....	....	+ 873.554	+ 930.854
I	2′ + 4	....	+37.149	+ 9.508	....	....	+ 832.126	+ 878.783
I′	$\text{I} \times -\frac{9.508}{37.149}$ *i. e.*, — 0.255942	....	[— 9.508]	— 2.433	....	....	— 212.976	— 224.918
6	....	....	[+ 9.508]	+38.284	+ 9.508	....	+1 393.927	+1 451.227
II	I′ + 6	....	....	+35.851	+ 9.508	....	+1 180.951	+1 226.309
II′	$\text{II} \times -\frac{9.508}{35.851}$ *i. e.*, — 0.265209	....	....	[— 9.508]	— 2.522	....	— 313.199	— 325.228
8	....	....	....	[+ 9.508]	+38.284	+ 9.508	+1 920.151	+1 977.451
III	II′ + 8	....	....	....	+35.762	+ 9.508	+1 606.952	+1 652.223
III′	$\text{III} \times -\frac{9.508}{35.762}$ *i. e.*, — 0.265869	....	....	....	[— 9.508]	— 2.528	— 427.238	— 439.275
10	....	....	....	....	[+ 9.508]	+79.649	+2 909.092	+2 998.249
IV	III′ + 10	....	....	....	....	+77.121	+2 481.854	+2 558.974
		—2.3948	—16.43813	—23.29260	—36.37862	—32.18130		

Equation (13) gives:

$$+0.296\ \tau_{13} - 8.779 - 0.037 \times 18.25 - 0.037 \times 36.50 = 0,$$

that is, $+0.296\ \tau_{13} - 10.805 = 0,$

$$\tau_{13} = +86.50.$$

Equation (14) gives:

$$+0.296\ \tau_{14} - 7.428 - 0.037 \times 91.24 = 0,$$
$$\tau_{14} = +36.50.$$

Equation (15) gives:

$$+0.296\ \tau_{15} - 16.877 - 0.037 \times 127.72 = 0.$$
$$\tau_{15} = +72.98.$$

Equation (16) gives:

$$+0.296\ \tau_{16} - 17.552 - 0.037 \times 109.47 = 0,$$
$$\tau_{16} = +72.98.$$

TABLE 33.—SECONDARY STRESSES IN TOP LATERAL SYSTEM DUE TO DEAD AND UNSYMMETRICAL LIVE LOADS. VALUES OF MOMENTS.

2	τ_{22}'	= − 2.395	M_{22}'	= 121.118 (− 4.790 + 2.395) = − 290.08 in.-lb.
		......		
	$\tau_{2.13}$	= − 2.395	$M_{2.13}$	= 0.074 (− 4.790 + 0) = − 0.35
		+ 18.250		
	τ_{24}	= + 15.855	$M_{2.4}$	= 19.016 (+ 31.710 − 16.438) = + 290.41
13	$\tau_{13.4}$	= + 36.50	$M_{13.4}$	= 0.074 (+ 73.00 + 1.812) = + 5.54 in.-lb.
		− 36.50		
	$\tau_{13.2}$	=	$M_{13.2}$	= 0.074 (0 − 2.395) = − 0.18
		......		
	$\tau_{13.2}'$	=	$M_{13.2}'$	= + 0.18
		− 36.50		
	$\tau_{13.4}'$	= − 36.50	$M_{13.4}'$	= − 5.54
4	$\tau_{4.2}$	= − 16.438	M_{42}	= 19.016 (− 32.876 + 15.855) = − 323.67 in.-lb.
		+ 18.250		
	$\tau_{4.13}$	+ 1.812	$M_{4.13}$	= 0.074 (+ 3.624 + 36.50) = + 2.97
		......		
	τ_{44}'	= + 1.812	M_{44}'	= 0.208 (+ 3.624 − 1.812) = + 0.38
		......		
	$\tau_{4.14}$	= + 1.812	$M_{4.14}$	= 0.074 (+ 3.624 + ...) = + 0.27
		+ 18.250		
	τ_{46}	= + 20.062	$M_{4.6}$	= 19.016 (+ 40.124 − 23.293) = + 320.06
14	$\tau_{14.6}$	= + 36.50	$M_{14.6}$	= 0.074 (+ 73.00 − 5.043) = + 5.03 in.-lb.
		− 36.50		
	$\tau_{14.4}$	=	$M_{14.4}$	= 0.074 (.... + 1.812) = + 0.13
		......		
	$\tau_{14.4}'$	=	$M_{14.4}'$	= = − 0.13
		− 36.50		
	$\tau_{14.6}'$	= − 36.50	$M_{14.6}'$	= = − 5.03
6	τ_{64}	= − 23.293	M_{64}	= 19.016 (− 46.586 + 20.062) = − 504.38 in.-lb.
		+ 18.250		
	$\tau_{6.14}$	= − 5.043	$M_{6.14}$	= 0.074 (− 10.086 + 36.50) = + 1.95
		......		
	$\tau_{6.6}'$	= − 5.043	$M_{6.6}'$	= 0.208 (− 10.086 + 5.043) = − 1.05
		......		
	$\tau_{6.15}$	= − 5.043	$M_{6.15}$	= 0.074 (− 10.086 + ...) = − 0.75
		+ 36.490		
	$\tau_{6.8}$	= + 31.447	$M_{6.8}$	= 19.016 (+ 62.894 − 36.379) = + 504.21
15	$\tau_{15.8}$	= + 72.98	$M_{15.8}$	= 0.074 (+ 145.96 + 0.111) = + 10.81 in.-lb.
		− 72.98		
	$\tau_{15.6}$	=	$M_{15.6}$	= 0.074 (.... − 5.043) = − 0.37
		......		
	$\tau_{15.6}'$	=	$M_{15.6}'$	= + 0.37
		− 72.98		
	$\tau_{15.8}'$	= − 72.98	$M_{15.8}'$	= − 10.81
8	τ_{86}	= − 36.379	$M_{8.6}$	= 19.016 (− 72.758 + 31.447) = − 785.57 in.-lb.
		+ 36.490		
	$\tau_{8.15}$	= + 0.111	$M_{8.15}$	= 0.074 (+ 0.222 + 72.98) = + 5.42
		......		
	τ_{88}'	= + 0.111	$M_{8.8}'$	= 0.208 (+ 0.222 − 0.111) = + 0.02
		......		
	$\tau_{8.16}$	= + 0.111	$M_{8.16}$	= 0.074 (+ 0.222 + ...) = + 0.02
		+ 36.490		
	$\tau_{8.10}$	= + 36.601	$M_{8.10}$	= 19.016 (+ 73.202 − 32.181) = + 780.06
16	$\tau_{16.10}$	= + 72.98	$M_{16.10}$	= 0.074 (+ 145.96 + 4.309) = + 11.12 in.-lb.
		− 72.98		
	$\tau_{16.8}$	=	$M_{16.8}$	= 0.074 (.... + 0.111) = + 0.01
		......		
	$\tau_{16.8}'$	=	$M_{16.8}'$	= − 0.01
		− 72.98		
	$\tau_{16.10}'$	= − 72.98	$M_{16.10}'$	= − 11.12
10	$\tau_{10.8}$	= − 32.181	$M_{10.8}$	= 19.016 (− 64.362 + 36.601) = − 527.90 in.-lb.
		+ 36.490		
	$\tau_{10.16}$	= + 4.309	$M_{10.16}$	= 0.074 (+ 8.618 + 72.980) = + 6.04
		......		
	$\tau_{10.10}$	= + 4.309	$M_{10.10}'$	= 121.118 (+ 8.618 − 4.309) = + 521.90

As the angles at the joints remain right angles at all times, because of the rigidity of the joints in the Plane 6-6′-7′-7, the posts suffer bending at their lower ends when the floor-beam is loaded. The shape of the frame is then somewhat like the dotted part of Fig. 74.

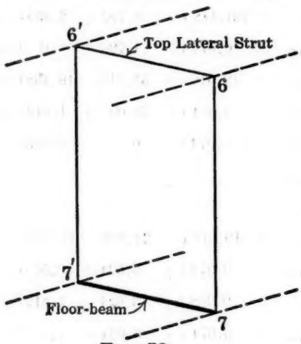


Fig. 73.

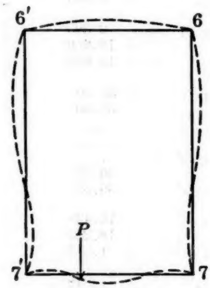


Fig. 74.

It must be remembered that no direct stresses enter the frame, for all the loads are taken care of by the diagonals and bottom chords that meet in Joints (7) and (7′). The analyses of Table 2 show that the primary stresses in Members 6-7 and 6′-7′ for dead and live loads are always zero.

There are three ways of attacking the problem, namely, the method of least work, Mohr's elastic weight method, and Batho's slope-deflection method. The first method will be omitted as it is unnecessarily long.

Mohr's Method.—The principles underlying this method are exactly those of the method for trusses, already described (see page 999), and, therefore, need no explanation. E is again regarded as unity.

From the dimensions of the rectangular frame, Fig. 75,

$$I_{6'\text{-}6} = 529 \text{ in.}^4$$

and

$$l_{6'\text{-}6} = 204 \text{ in.} \therefore \beta_{6'\text{-}6} = \frac{204}{2 \times 529} = 0.1928. \text{ (See Equation (78))}$$

$$I_{6'\text{-}7'} = I_{6\text{-}7} = 548 \text{ in.}^4$$

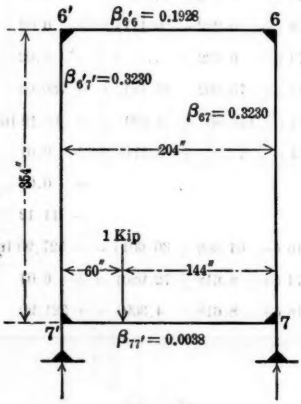


Fig. 75.

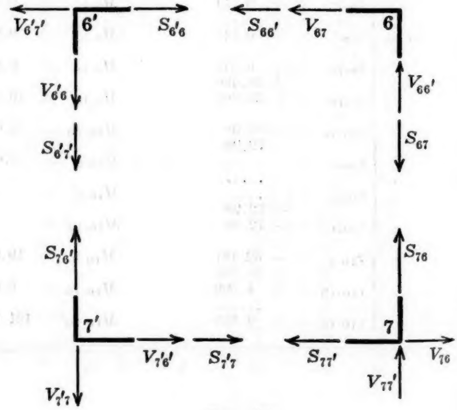


Fig. 76.

and,

$$l_{6'\text{-}7'} = l_{6\text{-}7} = 354 \text{ in.} \therefore \beta_{6'\text{-}7'} = \beta_{6\text{-}7} = \frac{354}{2 \times 548} = 0.3230$$

$$I_{7\text{-}7'} = 27\ 020 \text{ in.}^4$$

and,

$$l_{7\text{-}7'} = 204 \text{ in.} \therefore \beta_{7\text{-}7'} = \frac{204}{2 \times 27\ 020} = 0.0038$$

If the equilibrium of the joints is considered, as represented in Fig. 76, certain shearing forces, V, are induced at the joints due to the end bending moments resulting from the rigidity of the joints. These bending moments, eight in all, are the unknowns to be determined. $M_{6'\,7'}$ and $M_{7'\,6'}$ of Member 6'-7' give rise to the two equal and opposite shearing forces, $V_{6'\,7'}$ and $V_{7'\,6'}$ at Joint (6') and Joint (7'), respectively. In the sense that they are drawn, positive bending moments are calculated clockwise (see Fig. 5). All the other V's are inserted in the same manner. Finally, internal direct stresses, S, are induced.

For equilibrium of joints under direct stresses:

$$\left.\begin{array}{llll} (a)\ S_{6'\text{-}7'} = +\ V_{7\text{-}7'} & ; & (b)\ S_{6'\text{-}7'} = -\ V_{6'\text{-}6} \\ (c)\ S_{6'\text{-}6} = +\ V_{7'\text{-}6'} & ; & (d)\ S_{6'\text{-}6} = -\ V_{6\text{-}7} \\ (e)\ S_{6\text{-}7} = +\ V_{6'\text{-}6} & ; & (f)\ S_{6\text{-}7} = -\ V_{7\text{-}7'} \\ (g)\ S_{7\text{-}7'} = +\ V_{6\text{-}7} & ; & (h)\ S_{7\text{-}7'} = -\ V_{7'\text{-}6'} \end{array}\right\} \quad \ldots\ldots\ldots\ldots (101)$$

For equilibrium of the joints under bending moments:

$$\left.\begin{array}{l} (a)\ M_{7'\text{-}7} = -\ M_{7'\text{-}6'} \\ (b)\ M_{6'\text{-}7'} = -\ M_{6'\text{-}6} \\ (c)\ M_{6\text{-}6'} = -\ M_{6\text{-}7} \\ (d)\ M_{7\text{-}6} = -\ M_{7\text{-}7'} \end{array}\right\} \quad \ldots\ldots\ldots\ldots (102)$$

Also,

$$\left.\begin{array}{l} (a)\ V_{7'\text{-}6'} = V_{6'\text{-}7'} = \dfrac{M_{7'\text{-}6'} + M_{6'\text{-}7'}}{354} \\ (b)\ V_{6'\text{-}6} = V_{6\text{-}6'} = \dfrac{M_{6'\text{-}6} + M_{6\text{-}6'}}{204} \\ (c)\ V_{6\text{-}7} = V_{7\text{-}6} = \dfrac{M_{6\text{-}7} + M_{7\text{-}6}}{354} \\ (d)\ V_{7\text{-}7'} = V_{7'\text{-}7} = \dfrac{M_{7\text{-}7'} + M_{7'\text{-}7}}{204} \end{array}\right\} \quad \ldots\ldots\ldots\ldots (103)$$

This is easily proved as follows: Let two end moments, M_{ab}, M_{ba}, act on Member a-b, Fig. 77, and let the two end shears be V_{ab} and V_{ba}.

For equilibrium in a vertical direction, $V_{ab} = V_{ba}$. Also, $M_x = M_{ab} - V_{ab} \cdot x$.

When $x = l$,

$$M_{ba} + M_{ab} - V_{ab} \cdot l = 0$$

or,

$$V_{ab} = V_{ba} = \frac{M_{ab} + M_{ba}}{l}$$

Consider now the elastic weights that are involved in the problem (see Fig. 79):

$$\left.\begin{array}{ll}
(a)\ \phi_{7'\text{-}6'} = +\,\beta_{7'\text{-}6'}\,.\,M_{7'\text{-}6'} = & +\ 0.3230\ M_{7'\text{-}6'} \\
(b)\ \phi_{6'\text{-}7'} = -\,\beta_{7'\text{-}6'}\,.\,M_{6'\text{-}7'} = & -\ 0.3230\ M_{6'\text{-}7'} \\
(c)\ \phi_{6'\text{-}6} & +\ 0.1928\ M_{6'\text{-}6} \\
(d)\ \phi_{6\text{-}6'} & -\ 0.1928\ M_{6\text{-}6'} \\
(e)\ \phi_{6\text{-}7} & +\ 0.3230\ M_{6\text{-}7} \\
(f)\ \phi_{7\text{-}6} & -\ 0.3230\ M_{7\text{-}6} \\
(g)\ \phi_{7\text{-}7'} & +\ 0.0038\ M_{7\text{-}7'} \\
(h)\ \phi_{7'\text{-}7} & -\ 0.0038\ M_{7'\text{-}7}
\end{array}\right\} \quad \ldots\ldots\ldots\ldots (104)$$

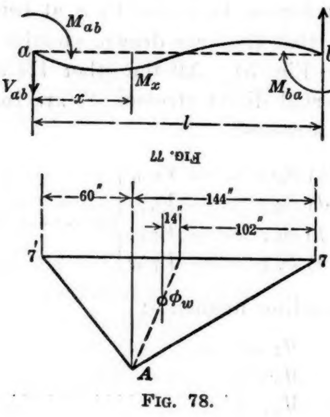


FIG. 78.

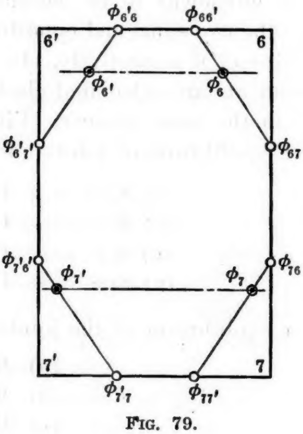


FIG. 79.

These eight quantities may be replaced by four elastic weights, the positions of which are determined by:

$$\left.\begin{array}{l}
\phi_{7'\text{-}6'}\ \phi_{7'}\ :\ \phi_{7'}\ \phi_{7'\text{-}7}\ =\ 0.0038\ :\ 0.3230 \\
\phi_{6'\text{-}7'}\ \phi_{6'}\ :\ \phi_{6'}\ \phi_{6'\text{-}6}\ =\ 0.1928\ :\ 0.3230 \\
\phi_{6\text{-}6'}\ \ \phi_{6}\ \ :\ \phi_{6}\ \ \phi_{6\text{-}7}\ \ =\ 0.3230\ :\ 0.1928 \\
\phi_{7\text{-}6}\ \ \phi_{7}\ \ :\ \phi_{7}\ \ \phi_{7\text{-}7'}\ \ =\ 0.0038\ :\ 0.3230
\end{array}\right\} \quad \ldots\ldots\ldots\ldots (105)$$

and the weights of which are:

$$\left.\begin{array}{l}
\phi_{7'} = +\ 0.3268\ M_{7'\text{-}6'} \\
\phi_{6'} = +\ 0.5158\ M_{6'\text{-}6} \\
\phi_{6} = +\ 0.5158\ M_{6\text{-}7} \\
\phi_{7} = +\ 0.3268\ M_{7\text{-}7'}
\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (106)$$

The four quantities are at the corners of a trapezoid the parallel sides of which, $\phi_{6'}$-ϕ_6 and $\phi_{7'}$-$\phi_{7'}$, have the lengths.

$$\phi_{6'}\text{-}\phi_6 = 68 + \frac{3230}{5158} \times 136 = 153.16 \text{ in.}$$

and,

$$\phi_{7'}\text{-}\phi_7 = 68 + \frac{3230}{3268} \times 136 = 202.42 \text{ in.}$$

also, the height of which is:

$$+ 354 = \frac{3230}{3268} \times 118 - \frac{3230}{5158} \times 118 = 163.48 \text{ in.}$$

To account for the elastic weight due to the bending moment of 1 kip on Member 7-7′, 60 in. from Point (7′) (see Fig. 78), the bending moment area is a triangle, the height of which is $\frac{1\ 000 \times 60 \times 144}{204}$ in-lb.

The area, therefore, is

$$\frac{1}{2} \times 204 \times \frac{1\ 000 \times 60 \times 144}{204} = (12\ 000 \times 360) \text{ in-sq-lb.}$$

As,

$$I_{7\text{-}7'} = 27\ 020 \text{ in.}$$

$$\phi_w = + \frac{12\ 000 \times 360}{27\ 020} = + 159.88$$

The point of application of ϕ_w is in Member 7′-7, distant 88 in. from Point (7′), this being the distance of the center of gravity of $\Delta\, 7'\, A\, 7$ from Point (7′).

The elastic chain then is replaced by the system of weights shown in Fig. 80.

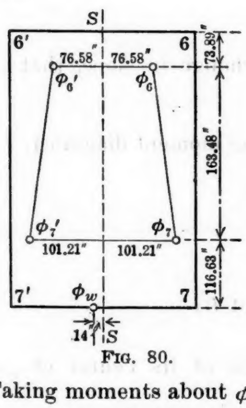


FIG. 80.

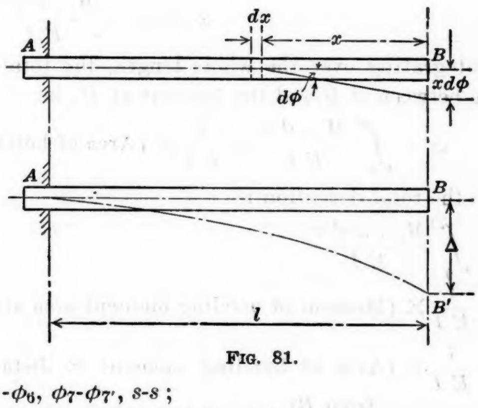


FIG. 81.

Taking moments about $\phi_{6'}$-ϕ_6, ϕ_7-$\phi_{7'}$, s-s;

$$+ (0.3268\ M_{7'\text{-}6'} \times 163.48) + (0.\ 3268\ M_{7\text{-}7'} \times 163.48) + (159.88 \times 280.11) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (107)$$

$$+ (0.5158\ M_{6'\text{-}6} \times 163.48) + (5\ 158\ M_{6\text{-}7} \times 163.48) - (159.88 \times 116.63) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (108)$$

$$+ (0.5158\ M_{6'\text{-}6} \times 76.58) - (0.5158\ M_{6\text{-}7} \times 76.58) + (0.3268\ M_{7'\text{-}6'} \times 101.21) - (0.3268\ M_{7\text{-}7'} \times 101.21) + 159.88 \times 14 = 0 \ldots\ldots\ldots\ldots (109)$$

Also, Equations (101) and (102) give

$$V_{7'\text{-}7} + V_{6'\text{-}6} = 0$$

or,

$$\frac{M_{7\text{-}7'} + M_{7'\text{-}7}}{204} + \frac{M_{6'\text{-}6} + M_{6\text{-}6'}}{204} = 0$$

that is,

$$M_{7\text{-}7'} + M_{7'\text{-}7} + M_{6'\text{-}6} + M_{6\text{-}6'} = 0$$

that is,

$$M_{7\text{-}7'} - M_{7'\text{-}6} + M_{6'\text{-}6} - M_{6\text{-}7} = 0 \dots\dots\dots\dots\dots (110)$$

The four equations, (107), (108), (109), and (110), contain the four unknowns, $M_{7'\text{-}6}$, $M_{6'\text{-}6}$, $M_{6\text{-}7}$, and $M_{7\text{-}7'}$ only. When these equations are solved:

$$\left.\begin{aligned} M_{7'\text{-}6'} &= -\ 434.55 \text{ in-lb.} \\ M_{6'\text{-}6} &= +\ \ 95.15 \text{ in-lb.} \\ M_{6\text{-}7} &= +\ 125.99 \text{ in-lb.} \\ M_{7\text{-}7'} &= -\ 403.71 \text{ in-lb.} \end{aligned}\right\} \dots\dots\dots\dots\dots (111)$$

The four remaining bending moments are the reciprocals of these, respectively.

Batho's Method.—This is a very elegant method, depending on two well-known properties of a beam subjected to bending. By referring to page 999, the relationship between the "elastic weight" idea and the following will be evident: Let $A\ B$ be a beam (Fig. 81), subjected to bending. As before, Equation (76),

$$d\,\phi = \frac{M_x\,.\,d_x}{E\,I}$$

while the contribution to the deflection from B is,

$$x\,.\,d\,\phi = \frac{M_x\,.\,x\,d\,x}{E\,I}$$

Integrating over the whole length, the total change in slope, that is, the angle between $A\ B$ and the tangent at B', is:

$$\phi = \int_0^l \frac{M_x\,.\,d\,x}{E\,I} = \frac{1}{E\,I} \times \text{(Area of bending moment diagram)} \dots (112)$$

while the total deflection is:

$$\begin{aligned} \Delta &= \int_0^l \frac{M_x\,.\,x\,d\,x}{E\,I} \\ &= \frac{1}{E\,I} \times \text{(Moment of bending moment area about } B) \\ &= \frac{1}{E\,I} \times \text{(Area of bending moment} \times \text{distance of its center of gravity from } B) \dots\dots (113) \end{aligned}$$

Consider a complete rectangular frame, Fig. 82. Imagine the top member, $A\ B$, to be broken at its mid-point, O, and let a bending moment, M, a horizontal thrust, H, and a vertical shear, V, act on the end, O of $A\ O$, while the reverse of these act on the end, O, of $B\ O$.

Now, consider the effect of the M, H and V, on End O of $A\ O$ separately.

The bending moment diagram for the M is as shown in Fig. 83. Imagine the frame to be resting on two hinges at D and C. The bending moment is constant over the portion, O-A-D, but between D and C, it varies uniformly from $+M$ to zero, while no bending occurs in $C\ B$ or $B\ O$. Now, determine the amount of movement, x, to the left that the point, O, undergoes. The bending moment area over $A\ O$ contributes a quantity of the second order of smallness. From Fig. 81, it is clear that a small displacement occurs perpendicularly to the member, while the movement in the direction of the

member is negligible. Now, the bending moment along $A\,D$ causes A to deflect to the left from its initial position by an amount equal to (see Equation (113)):

$$\frac{1}{E\,I_2} \times M\,h \times \frac{h}{2} = \frac{M\,h^2}{2\,E\,I_2}$$

but in doing so, it carries $A\,O$, and, therefore, the point, O, to the left by this amount.

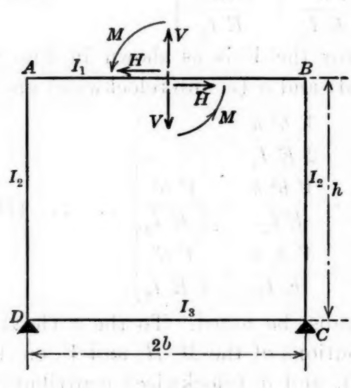


FIG. 82.

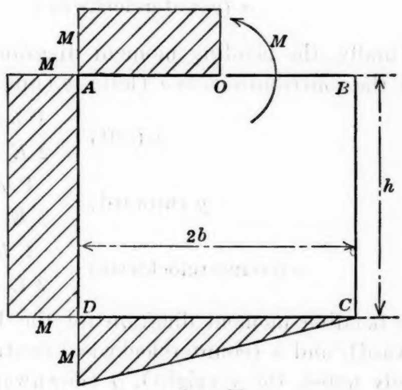


FIG. 83.

The bending moments along $D\,C$ causes a change in slope between the end tangents at D and C equal to (see Equation (112)):

$$\frac{1}{E\,I_3} \times \frac{1}{2}\,M\,.\,2\,b = \frac{M\,b}{E\,I_3}$$

but as the angle between the tangents to $D\,A$ and $D\,C$ is constant, $D\,A$ is deflected to the left by an amount equal to $\dfrac{M\,b}{E\,I_3}$. In doing so, A, and, therefore, O, is moved to the left by an amount equal to:

$$\frac{M\,b}{E\,I_3} \times h = \frac{M\,b\,h}{E\,I_3}$$

The total contribution of M to the horizontal movement of O, to the left is, therefore,

$$\frac{M\,h^2}{2\,E\,I_2} + \frac{M\,b\,h}{E\,I_3}$$

It is possible to find similar expressions for the upward (y) displacement of O from its initial position, and for the total rotation between the end tangent at O of $A\,O$ and the initial position, $A\,O$ (counterclockwise). These expressions are:

$$\left.\begin{aligned} x\text{ (left)} &= \frac{M\,h^2}{2\,E\,I_2} + \frac{M\,b\,h}{E\,I_3} \\ y\text{ (upward)} &= \frac{M\,b^2}{2\,E\,I_1} + \frac{M\,h\,b}{E\,I_2} + \frac{M\,b^2}{E\,I_3} \\ \alpha\text{ (counterclockwise)} &= \frac{M\,b}{E\,I_1} + \frac{M\,h}{E\,I_2} + \frac{M\,b}{E\,I_3} \end{aligned}\right\} \quad\ldots\ldots\ldots(114)$$

Similarly, the bending moment diagram for H is as shown in Fig. 84, while the contributions to x (left), y (upward) and α (counterclockwise) are :

$$\left.\begin{aligned} x \text{ (left)} &= \frac{H\,h^3}{3\,E\,I_2} + \frac{H\,h^2\,b}{E\,I_3} \\ y \text{ (upward)} &= \frac{H\,h^2\,b}{2\,E\,I_2} + \frac{H\,h\,b^2}{E\,I_3} \\ \alpha \text{ (counterclockwise)} &= \frac{H\,h^2}{2\,E\,I_2} + \frac{H\,h\,b}{E\,I_3} \end{aligned}\right\} \ldots\ldots\ldots\ldots (115)$$

Finally, the bending moment diagram for the V is as shown in Fig. 85, while the contributions to x (left), y (upward) and α (counterclockwise) are:

$$\left.\begin{aligned} x \text{ (left)} &= \frac{V\,b\,h^2}{2\,E\,I_2} + \frac{V\,b^2\,h}{2\,E\,I_3} \\ y \text{ (upward)} &= \frac{V\,b^3}{3\,E\,I_1} + \frac{V\,b^2\,h}{E\,I_2} + \frac{V\,b^3}{2\,E\,I_3} \\ \alpha \text{ (counterclockwise)} &= \frac{V\,b^2}{2\,E\,I_1} + \frac{V\,b\,h}{E\,I_2} + \frac{V\,b^2}{2\,E\,I_3} \end{aligned}\right\} \ldots\ldots\ldots (116)$$

(The bending moment diagram for the V must be noted. To the x (left), y (upward), and α (counterclockwise) contributions of the M, H, and V, on AO, already noted, the x (right), y (downward), and α (clockwise) contributions

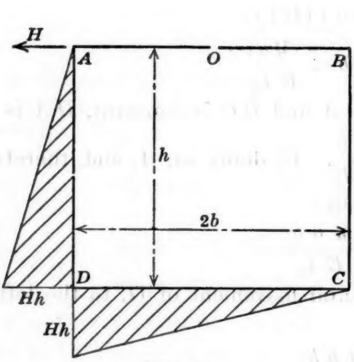


FIG. 84.

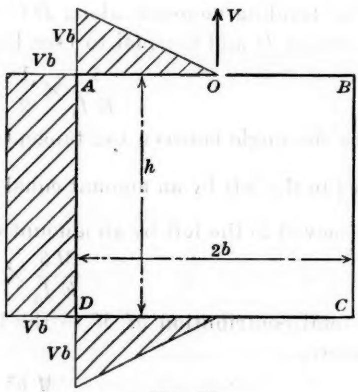


FIG. 85.

of the M, H, and V, on BO must be added. As the two V's act in opposite directions, the bending moment at the center point of DC due to the V's will be zero, and the portion shown on DC of Fig. 85 may be regarded as that due to the V on AO.) Consider now the combined effects of the two sets of M, H, and V. Notice that the M on AO, together with the M on BO, doubles the x-distance between the ends at O, and also doubles the α-angle between the end tangents at O, the respective ends being moved an equal vertical amount; in other words, the relative vertical displacement is zero. Similarly, the two H's double the x-contribution, and also the α-contribution, but the y-contribution becomes zero.

On the other hand, the two V's give zero contributions to the x and the α, but double the y-contribution. The combination of these is indicated as follows:

	x	y	α
M	2	0	2
H	2	0	2
V	0	2	0

Hence, if the top member is cut, the ends become separated under the two sets of M, H, and V, so that:

$$\left.\begin{aligned} x &= \frac{M\,h^2}{E\,I_2} + \frac{2\,M\,b\,h}{E\,I_3} + \frac{2\,H\,h^3}{3\,E\,I_2} + \frac{2\,H\,h^2\,b}{E\,I_3} \\ y &= \frac{2\,V\,b^3}{3\,E\,I_1} + \frac{2\,V\,b^2\,h}{E\,I_2} + \frac{V\,b^3}{E\,I_3} \\ \alpha &= \frac{2\,M\,b}{E\,I_1} + \frac{2\,M\,h}{E\,I_2} + \frac{2\,M\,b}{E\,I_3} + \frac{H\,h^2}{E\,I_2} + \frac{2\,H\,h\,b}{E\,I_3} \end{aligned}\right\} \qquad \ldots\ldots\ldots (117)$$

The x is the amount by which the O's of $A\,O$ and $B\,O$ are separated horizontally, the O of $A\,O$ lying to the left and the O of $B\,O$ to the right. The y is the vertical gap, the O of $A\,O$ lying above the O of $B\,O$. The α is the angle between the end tangents, the $A\,O$ being turned counterclockwise and the $B\,O$, clockwise.

Next, determine the effect on the two ends of the cut frame under the actual load, instead of the M, H, and V. The x must now be the horizontal distance between the ends, but with this difference that $A\,O$ and $B\,O$ overlap. The y must be the vertical distance between the ends, but the $B\,O$ must now lie above $A\,O$. The α must be the angle between the end tangents, but the $A\,O$ is now considered as being turned clockwise while the $B\,O$ is turned counterclockwise.

Equating the corresponding x's, y's, and α's, of the M-H-V-set and the load-set, gives three linear equations for the three unknowns M, H, and V. The values of M, H, and V are then those that actually exist at the center of the top member of the uncut frame.

Before applying the theory to the cross-frame, Members 6-6′-7′-7, the formulas for the deflection angles at the ends of a simply supported beam, when it is subjected to a single concentrated load, will be reproduced.

The deflection angles, α_1 and α_2*, of the beam shown in Fig. 86 are given by:

$$\left.\begin{aligned} \alpha_1 &= \frac{P}{12\,E\,I\,b}\,(p^3 - 6\,p^2\,b + 8\,p\,b^2) \\ \alpha_2 &= \frac{P}{12\,E\,I\,b}\,(4\,p\,b^2 - p^3) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots (118)$$

* Church's "Mechanics", Article (234), Equation (7).

Turning now to the problem of the cross-frame, 6-6′-7′-7, shown again in Fig. 88, the α, x, and y-terms from the kip load are:

$$\alpha = \alpha_1 + \alpha_2 = \frac{P}{12\, I_3\, b}\,(12\, p\, b^2 - 6\, p^2\, b)$$

$$= \frac{1\,000}{12 \times 27\,020 \times 102}\,(12 \times 60 \times 102^2 - 6 \times 60^2 \times 102)$$

$$x = \alpha_1\, h + \alpha_2\, h = \frac{P}{121_3\, b}\,(12\, p\, b^2 - 6\, p^2\, b)\, h$$

$$= \frac{1\,000}{12 \times 27\,020 \times 102}\,(12 \times 60 \times 102^2 - 6 \times 60^2 \times 102) \times 354$$

and,

$$y = \alpha_1\, b - \alpha_2\, b = \frac{P}{121_3}\,(2\, p^3 - 6\, p^2\, b + 4\, p\, b^2)$$

$$= \frac{1\,000}{12 \times 27\,020}\,(2 \times 60^3 - 6 \times 60^2 \times 102 + 4 \times 60 \times 102^2).$$

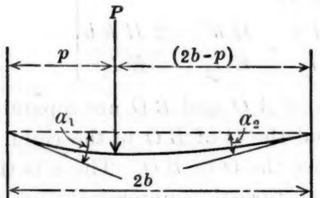


FIG. 86

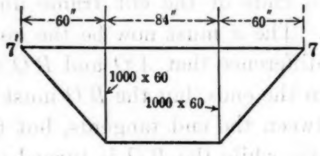


FIG. 87.

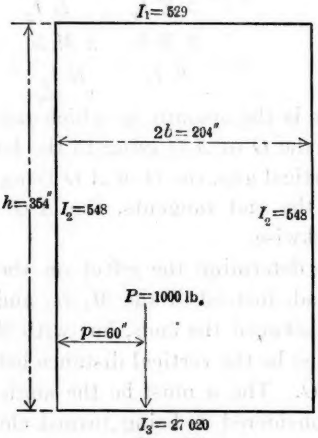


FIG. 88.

These reduce to:

$$\alpha = \frac{36 \times 12 \times 1\,000}{2\,702}$$

$$x = \frac{354 \times 36 \times 12 \times 1\,000}{2\,702}$$

and,

$$y = \frac{6\,048 \times 1\,000}{2\,702}$$

Making use of Equations (117):

$$\frac{M \times 354^2}{548} + \frac{2\, M \times 102 \times 354}{27\,020} + \frac{2\, H \times 354^3}{3 \times 548}$$

$$+ \frac{2\, H \times 354^2 \times 102}{27\,020} = \frac{354 \times 36 \times 12 \times 1\,000}{2\,702}$$

$$\frac{2\, V \times 102^2 \times 354}{548} + \frac{2\, V \times 102^3}{3 \times 529} + \frac{V \times 102^3}{27\,020} = \frac{6\,048 \times 1\,000}{2\,702}$$

and,

$$\frac{2\,M \times 354}{548} + \frac{2\,M \times 102}{529} + \frac{2\,M \times 102}{27\,020} + \frac{H \times 354^2}{548}$$
$$+ \frac{2\,H \times 354 \times 102}{27\,020} = \frac{36 \times 12 \times 1\,000}{2\,702}$$

giving,

$$V = +\,0.151 \text{ lb.}$$
$$H = +\,1.497 \text{ lb.}$$

and,

$$M = -\,110.579 \text{ in-lb.}$$

The bending moment diagram is, therefore, as shown in Fig. 89. Note how closely the values found by the two methods correspond.

	Mohr.	Batho.	Sense.
$M_{7'6'}$	434.55	437.32	Counterclockwise
$M_{6'6}$	95.15	92.45	Clockwise
$M_{6\,7}$	125.99	128.71	Clockwise
$M_{7\,7'}$	403.71	401.06	Counterclockwise

Thus far, the problem has been for an unsymmetrical load. The method will be useful in any subsequent work, where wind loads, causing an unsymmetrical load on the rails, are considered.

The present investigation, however, aims at symmetrical loads, that is, at the effect of a kip load on each rail, in the plane of the frame. Turning to Mohr's method, the bending moment area, Fig. 87, of the loads now becomes:

$$(60 \times 1\,000 \times 60) + (84 + 1\,000 \times 60) = 144 \times 60 \times 1\,000$$

$$\phi_w = \frac{144 \times 60 \times 1\,000}{27\,020} = +\,319.76$$

and acts at the center of the floor-beam, from symmetry.

Taking moments about the three axes used previously:

$$+ (0.3268\ M_{7'6'} \times 163.48) + (0.3268\ M_{7\,7'} \times 163.48) + 319.76 \times 280.11 = 0$$
$$+ (0.5158\ M_{6'6} \times 163.48) + (0.5158\ M_{6\,7} \times 163.48) - 319.76 \times 116.63 = 0$$
$$+ (0.5158\ M_{6'6} \times 76.58) - (0.5158\ M_{6\,7} \times 76.58) + (0.3268\ M_{7'6'} \times 101.21)$$
$$- (0.3268\ M_{7\,7'} \times 101.21) = 0$$

and, as before:

$$M_{7\text{-}7'} - M_{7'\text{-}6'} + M_{6'\text{-}6} - M_{6\text{-}7} = 0$$

The last two equations give:

$$33.075\ (M_{7'\text{-}6'} - M_{7\text{-}7'}) + 39.4999\ (M_{6'\text{-}6} - M_{6\text{-}7}) = 0$$

and,

$$(M_{7'\text{-}6'} - M_{7\text{-}7'}) - (M_{6'\text{-}6} - M_{6\text{-}7}) = 0$$

hence,

$$M_{7'\text{-}6'} = M_{7\text{-}7'} \text{ and } M_{6'\text{-}6} = M_{6\text{-}7}$$

which is to be expected from symmetry.

Hence,

$$M_{7'\text{-}6'} = -\ 838.25 \text{ in-lb.}$$

and,

$$M_{6'\text{-}6} = +\ 221.14 \text{ in-lb.}$$

(Note that the effect of a kip load on the rail near Joint (7) is just the reverse of that of a kip load on the rail near Joint (7′). The two values just found could easily have been reached by compounding $M_{7'\text{-}6'}$ and $M_{7\text{-}7'}$ of the unsymmetrical case, giving,

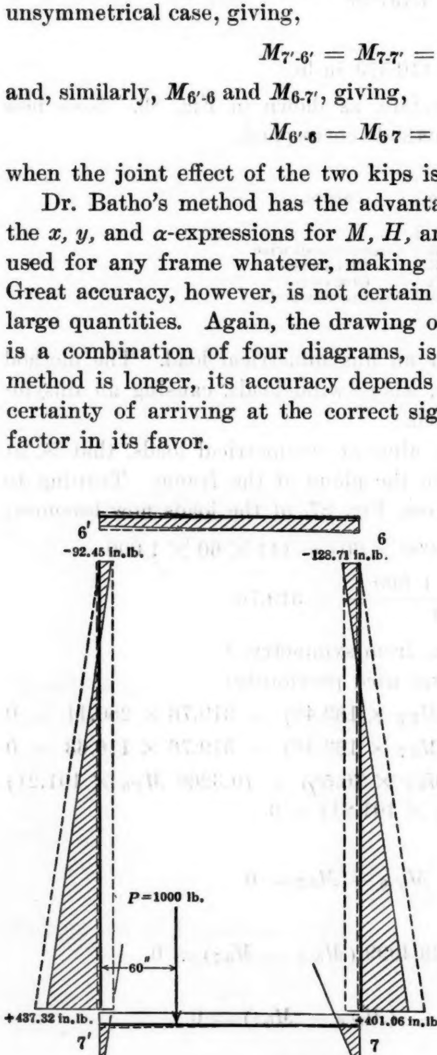
$$M_{7'\text{-}6'} = M_{7\text{-}7'} = -\ 838.25 \text{ in-lb.}$$

and, similarly, $M_{6'\text{-}6}$ and $M_{6\text{-}7'}$, giving,

$$M_{6'\text{-}6} = M_{6\ 7} = +\ 221.14 \text{ in-lb.}$$

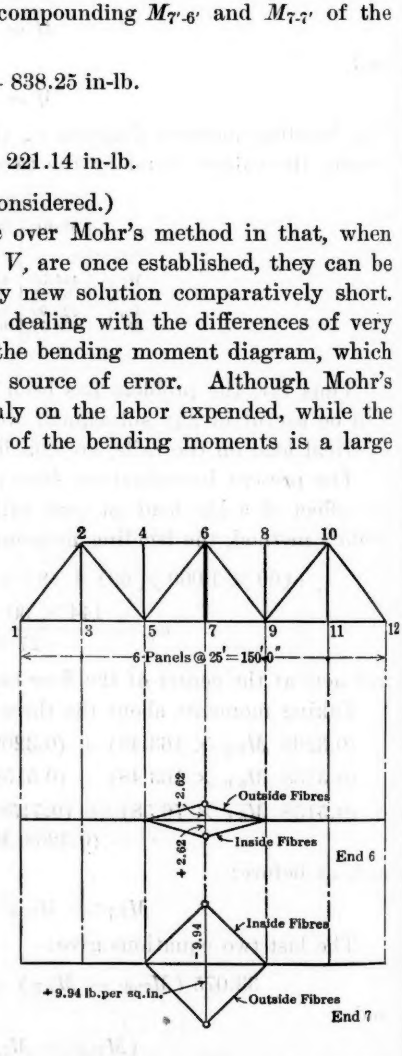
when the joint effect of the two kips is considered.)

Dr. Batho's method has the advantage over Mohr's method in that, when the x, y, and α-expressions for M, H, and V, are once established, they can be used for any frame whatever, making any new solution comparatively short. Great accuracy, however, is not certain in dealing with the differences of very large quantities. Again, the drawing of the bending moment diagram, which is a combination of four diagrams, is a source of error. Although Mohr's method is longer, its accuracy depends only on the labor expended, while the certainty of arriving at the correct signs of the bending moments is a large factor in its favor.



MOMENTS IN CROSS-FRAME BY BATHO'S METHOD.

FIG. 89.



INFLUENCE LINES OF SECONDARY STRESSES INDUCED IN 6-7 AS PART OF CROSS-FRAME.

FIG. 90.

In drawing the influence line for the vertical post, 6-7, as part of the cross-frame, note that it is zero over the whole span, except the two adjacent panels. As the member is such that all the extreme fibers are 6.5 in. away from the neutral plane, the influence lines could best be drawn for secondary stresses, and not for bending moments.

The outside fibers at End (7) of Member 6-7 are in tension, while the outside fibers at End (6) are in compression. The factor by which the bending moments are to be multiplied is $\frac{C}{I} = \frac{6.5}{548}$, giving the quantities found in Fig. 90.

5.—Application of the Influence Lines to the Solution of the Bridge Under Actual Loads

The total dead load of the bridge is 360 000 lb., or 30 000 lb. per panel point. It is assumed that the live load is a train of Cooper's E-50 class. Further, the impact allowance for primary stresses will be $\frac{300}{300 + L}$ of the live load contribution, in which L is the length of the span that is loaded. All the secondary stresses that arise from the live load are to be increased by 50% to allow for impact and vibration. Two typical cases will be computed.

Member 6-7.—The influence lines (Figs. 66 and 90), together with Cooper's loading diagram give the stresses. The dead load effects are the following:

The primary stress $= + 41.07 \times 30 = + 1\,232.10$ lb. per sq. in.

The secondary stresses, due to the rigidity of the truss joints, are zero.

The secondary stresses, due to the rigidity of the cross-frame, give:

For End (6), outside fibers.... $-2.62 \times 30 = -\ 78.60$ lb. per sq. in.
For End (6), inside fibers.... $+2.62 \times 30 = +\ 78.60$ " " " "
For End (7), outside fibers.... $+9.94 \times 30 = + 298.20$ " " " "
For End (7), inside fibers.... $-9.94 \times 30 = - 298.20$ " " " "

The maximum primary stress due to live load occurs when Wheel (5) is opposite Panel Point (7). Then,

$$\text{Primary stress} = + (12.5 \times 4.928) + 25\,(16.428 + 24.642 + 32.856 + 41.070) + 16.25\,(26.285 + 18.071 + 8.214) = + 3\,790.76 \text{ lb. per sq. in.}$$

$$\text{Impact} = + 3\,790.76 \times \left(\frac{300}{300 + 50}\right) = + 3\,249.22 \text{ lb. per sq. in.}$$

The secondary stresses, due to rigidity of truss joints, are:

For End (6), right fibers.... $-1.5\,(147.22) = - 220.83$ lb. per sq. in.
For End (6), left fibers.... $+ 220.83$ " " " "
For End (7), right fibers.... $+ 221.22$ " " " "
For End (7), left fibres.... $- 221.22$ " " " "

The secondary stresses, due to rigidity of cross-frame, are:

For End (6), outside fibers.... $-1.5\,(243.14) = - 364.71$ lb. per sq. in.
For End (6), inside fibers.... $+ 364.71$ " " " "
For End (7), outside fibers.... $+ 1\,383.67$ " " " "
For End (7), inside fibers.... $- 1\,383.67$ " " " "

Hence, under the severest conditions for dead and live load combined, the total stresses at the eight end corner fibers, (Fig. 91) become:

$$f_{6a} = + (1\,232 + 3\,791 + 3\,249) + (-78.60 - 220.83 - 364.71)$$
$$= +\,7\,608 \text{ lb. per sq. in.}$$

and, similarly,

$$f_{6b} = +\ \ 8\,050 \text{ lb. per sq in.}$$
$$f_{6c} = +\ \ 8\,936 \text{ " " " "}$$
$$f_{6d} = +\ \ 8\,494 \text{ " " " "}$$
$$f_{7a} = +\,10\,175 \text{ " " " "}$$
$$f_{7b} = +\ \ 9\,733 \text{ " " " "}$$
$$f_{7c} = +\ \ 6\,369 \text{ " " " "}$$
$$f_{7d} = +\ \ 6\,811 \text{ " " " "}$$

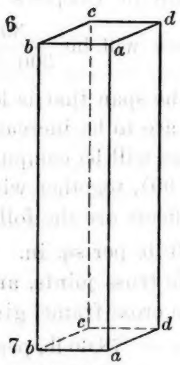


FIG. 91.

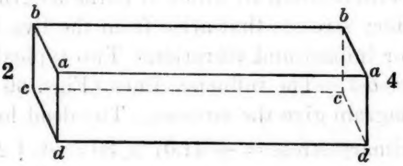


FIG. 92.

Member 2-4.—Referring to the influence lines (Figs. 59 and 71), the dead load gives:

Primary stress = — 2 233.20 lb. per sq. in.

The secondary stresses, due to rigidity of truss joints, give:

For End (2), top fibers.............. + 102.00 lb. per sq. in.
For End (2), bottom fibers........... — 104.40 " " " "
For End (4), top fibers.............. — 297.00 " " " "
For End (4), bottom fibers........... + 303.30 " " " "

The secondary stresses, due to the rigidity of the top lateral system, give:

For End (2), a bending moment area = + 49 049.25
For End (4), a bending moment area = — 66 307.05

These are to be multiplied by $\pm \dfrac{10.5}{2\,852.45}$ and $\pm \dfrac{12}{2\,852.45}$, giving,

For End (2), cover-plate outside fiber..... — 270.83 lb. per sq. in.
For End (2), cover-plate inside fiber..... + 270.83 " " " "
For End (2), bottom angles outside fiber... — 308.02 " " " "
For End (2), bottom angles inside fiber... + 308.02 " " " "

and,

For End (4), cover-plate outside fiber..... — 366.12 lb. per sq. in.
For End (4), cover-plate inside fiber...... + 366.12 " " " "
For End (4), bottom angle outside fiber.... — 418.42 " " " "
For End (4), bottom angle inside fiber.... + 418.42 " " " "

The maximum primary stress due to live load occurs when the bridge is almost fully loaded, namely, when Wheel (9) is at Panel Point (5). Then:

Primary stress = — 5 729.41 lb. per sq. in.,
Impact = — 3 819.61 " " " ".

The secondary stresses, due to the rigidity of the truss joints, become:

For End (2), bottom fibers.... — 1.5 (209.57) = — 314.35 lb. per sq. in.
For End (2), top fibers...... + 307.43 " " " "
For End (4), bottom fibers.... + 1 038.95 " " " "
For End (4), top fibers...... — 1 016.09 " " " "

The secondary stresses, due to the rigidity of the top lateral system, become:

For End (2), cover-plate outside fiber.......... — 693.39 lb. per sq. in.
For End (2), cover-plate inside fiber.......... + 693.39 " " " "
For End (2), bottom angle outside fiber........ — 792.44 " " " "
For End (2), bottom angle inside fiber........ + 792.44 " " " "
For End (4), cover-plate outside fiber.......... — 943.89 " " " "
For End (4), cover-plate inside fiber.......... + 943.89 " " " "
For End (4), bottom angle outside fiber...... — 1 078.74 " " " "
For End (4), bottom angle inside fiber....... + 1 078.74 " " " "

Hence, under the severest conditions, the total stresses at the eight corner fibers, Fig. (92), become:

$$f_{2a} = -(2\,233.20 + 5\,729.41 + 3\,819.61) + (102.00 - 270.83 + 307.43 - 693.39)$$
$$= -12\,337 \text{ lb. per sq. in.}$$
$$f_{2b} = -10\,409 \text{ " " " "}$$
$$f_{2c} = -11\,101 \text{ " " " "}$$
$$f_{2d} = -13\,301 \text{ " " " "}$$
$$f_{4a} = -14\,406 \text{ " " " "}$$
$$f_{4b} = -11\,785 \text{ " " " "}$$
$$f_{4c} = -8\,943 \text{ " " " "}$$
$$f_{4d} = -11\,937 \text{ " " " "}$$

6.—Conclusion

By means of the set of influence lines for the primary and secondary stresses in the truss members, due to the rigidity of the joints in the plane of the truss, and the sets of influence lines for the top lateral system and the cross-frames, it has been possible to determine the maximum stresses that occur at the end corners of all the members except the bottom chords. In order to obtain the maximum stresses in the bottom chords, it is first necessary to obtain the influence lines for the floor system. With such a complete set available, superimpose the secondary stresses due to dead load together with those brought

into play by the axial loads. At this point it would be well to use Westergaard's analysis (35). It must be remembered that the variation between two end moments in a certain plane follows the straight-line law, consequently, it is possible to obtain influence lines for secondary stresses due to the rigidity of the joints at any section of the member. All the influence lines that have been found thus far, are for sections at the joints. As end connections prevent the determination of end effects, and, also, as these connections result in stresses quite different from what they would be if members were supposed to be rigidly held at the very ends, observations are taken beyond the gusset-plates, etc. It is possible, however, to make an analysis for the secondary stresses at some such section and then verify them experimentally. Discrepancies are to be expected, but just as tension specimens have under the clamps a distribution of stress which is not uniform, and to all intents and purposes a uniform distribution a little beyond the clamps, so it may be inferred that, in practice, the theoretical stresses at the corner fibers of a member of a bridge will exist at sections beyond the end connections.

So much for the dead and the live load; consider the wind load separately. Its presence means certain horizontal loads at the panel points. An analysis of the top and bottom lateral systems will indicate the extra primary stresses in the chords, together with the additional secondary stresses in the two horizontal planes. The primary stresses induced by the wind in the top and bottom chords of each truss cause additional secondary stresses in these trusses. A set of influence lines for these must be drawn. Again, the overturning effect of the wind on the train throws more weight on the leeward rail than on the windward. Not only will the cross-frame secondary stresses due to the live load be effected, but, also, the secondary stresses in the two trusses caused by the live load. Imagine that the wind causes the load of 1 kip on the leeward rail to become 1.05 kips and that on the windward, 0.95 kip. With little effort the cross-frame secondary stresses may be determined from the analysis given on pages 1095 to 1111, while the secondary stress quantities found on pages 1076 to 1086 are to be multiplied by 1.05 to obtain those in the leeward truss and by 0.95 to obtain those in the windward truss. Finally, the top lateral system needs revision.

Traction effects should now be considered in more or less the same way. When all the influence lines have been determined, they should be combined in such a manner as to arrive at the maximum corner stress that a member can possibly have. It will generally be found to occur when the primary stress is a maximum. It seems justifiable, therefore, to compound the secondary stresses arising from axial and transverse loads at that moment with those already found from the end moments, and thus obtain a fairly good estimate of what happens along the length of the member.

The analysis given on pages 1111 to 1113, clearly shows how widely the four corner stresses at the end of a member may differ. Any device that lessens this difference is to be encouraged. In practice, the top members of the cross-frames are often longer than their theoretical length by an amount that introduces secondary stresses opposite in effect to those given on pages 1095 to 1111,

and equal to the dead load plus one-half the live load contribution. In large structures, too, either the top chords are intentionally made longer or the bottom chords shorter than their theoretical unstressed lengths, so that when the dead load acts, the top chords become straight; the axial effect, which is intensified with bent chords, is, therefore, minimized.

In conclusion, the writer believes that he has demonstrated conclusively, which methods the practical engineer must adopt in order best to obtain proper results. His idea of compounding secondary stresses as indicated is probably far from the truth, but it is an endeavor to explain, by means of the present limited knowledge of mathematics and mechanics, the true state of affairs in a bridge. He has indicated how this study can be pursued to a conclusion and, although he frankly acknowledges that it may never be adopted in its complete form in practice, he sincerely hopes that all engineers will regard such a study not so much as a possible means of material benefit but as a search for truth.

Acknowledgments

The writer wishes to express his thanks to H. M. MacKay, M. Am. Soc. C. E., and Dr. Cyril Batho, of McGill University, and to H. S. Jacoby, Affiliate, Am. Soc. C. E., and Dr. E. W. Rettger, of Cornell University, for their instruction, advice, and encouragement.

Bibliography

(1) **Winkler, E.** Die Lehre von der Elasticität und Festigkeit. Prag, 1867, p. 120.

(2) **Weyrauch, I. I.** Allgemeine Theorie und Berechnung der kontinuirlichen und einfachen Träger. Leipzig, 1873, p. 37.

(3) **Manderla, H.** Die Berechnung der Sekundarspannungen, welche im einfachen Fachwerke infolge starrer Knotenverbindungen auftreten. *Allgemeine Bauzeitung,* 1880.

(4) **Winkler, E.** Vorträge über Brückenbau, Theorie der Brücken, II Heft, XVI Kapitel: Einfluss fester Verbindungen. Wien, 1881, pp. 248-319.

(5) **Landsberg, Th.** Beitrag zur Theorie der Fachwerke. *Zeitschrift* des Arch.- und Ing.-Vereins zu Hannover, 1885, p. 362, and 1886, p. 195.

(6) **Müller-Breslau, H.** Zur Theorie der Biegungsspannungen in Fachwerkträgern. *Zeitschrift* des Arch.- und Ing.-Vereins zu Hannover, 1886, p. 399.

(7) **Ritter, W.** Anwendungen der graphischen Statik. Zurich, 1890, II Teil, pp. 171-208.

(8) **Engesser, Fr.** Die Zusatzkrafte und Nebenspannungen eiserner Fachwerkbrücken. Berlin, 1893, II Teil.

(9) **Mehrtens, G. Ch.** A Hundred Years of German Bridge Building. Translated by L. Mehrtens. Berlin, 1900, p. 45.

(10) **Patton, E.** Beitrag zur Berechnung der Nebenspannungen infolge starrer Knotenverbindungen bei Brückentragern. *Zeitschrift zur Architektur und Ingenieurwesen,* 1902.

(11) **Mehrtens, G. Ch.** Vorlesungen über Statik der Baukonstruktionen und Festigkeitslehre. Leipzig, 1905, Vol. III, pp. 380-402.

(12) **Hiroi, I.** The Statically-Indeterminate Stresses in Frames, Commonly Used for Bridges. New York, 1905.

(13) **Müller-Breslau, H.** Die graphische Statik der Baukonstruktionen und Festigkeitslehre. Second Edition. 1908, Vol. II, pp. 269-308.

(14) **Grimm, C. R.** Secondary Stresses in Bridge Trusses. New York, 1908.

(15) **Gehler, W.** Nebenspannungen eiserner Fachwerkbrücken. Berlin, 1910.

(16) **Kunz, F. C.** Secondary Stresses. *Engineering News,* Vol. 66, 1911, p. 397.

(17) **Vanderbeck, H. A.** An Experimental Study of the Secondary Stresses in Railway Bridges. A Thesis Presented for the Degree of M. C. E. from Cornell University. 1912.

(18) **Turneaure, F. E.** The Calculation of Secondary Stresses in Trusses. *Engineering News,* Vol. 68, 1912, p. 438.

(19) **Mohr, O.** Beitrag zur Theorie des Fachwerkes. *Der Eisenbau,* 1910, pp. 2 and 93.

Die Berechnung der Pfostenträger (Vierendeelträger). *Der Eisenbau,* 1912, p. 85.

Die Berechnung des Brückenrahmens. *Der Eisenbau,* 1912, p. 150.

Die Berechnung der Nebenspannungen in Fachwerken mit steifen Knotenverbindungen. *Der Eisenbau,* 1912, p. 181.

(20) **Pirlet.** Die Berechnung statisch unbestimmter Systeme. *Der Eisenbau,* 1910, p. 331.

Einflusslinien der Spannungen in Fachwerken mit starren Knotenpunktsverbindungen. *Der Eisenbau,* 1912, pp. 203 and 245.

(21) **Molitor, D. A.** Kinetic Theory of Engineering Structures. New York, 1911.

(22) **Thayer, H. R.** Structural Design. New York, 1914.

(23) **Kunz, F. C.** Design of Steel Bridges. New York, 1915.

(24) **Waddell, J. A. L.** Bridge Engineering. New York, 1916.

(25) **Johnson, Bryan, and Turneaure.** Modern Framed Structures. New York, 1916, Vol. II.

(26) **Merriman and Jacoby.** Roofs and Bridges. New York, 1919, Vol. III.

(27) **Páez, J.** A Comparative and Critical Study of the Different Methods of Computing Secondary Stresses in Bridge Trusses. A Thesis Presented for the Degree of M. C. E. from Cornell University. 1913.

(28) **Mao, T. E.** The Design of a Two-Hinged Spandrel-Braced Arch Bridge, with an Investigation of Its Secondary Stresses. A Thesis Presented for the Degree of M. C. E. from Cornell University. 1917.

(29) **American Railway Engineering Association.** *Proceedings,* 1914, p. 438, and 1916, p. 129.

(30) **Reichmann, Albert.** Notes on Structural Steel Designs. *Journal,* Western Soc. of Engrs., Vol. 17.

(31) **Pittman, E. W.** Secondary Stresses in Framed Structures. *Proceedings,* Engrs.' Soc. of Western Pennsylvania, Vol. 25.

(32) **Wilson, W. M.** Effects of Secondary Stresses on Design. *Journal,* Western Soc. of Engrs., Vol. 21.

(33) **Steinman, D. B.** Stress Measurement on the Hell Gate Arch Bridge. *Transactions,* Am. Soc. C. E., Vol. LXXXII (1918), p. 1040.

(34) **Mao, T. E.** Secondary Stresses. A Thesis Presented for the Degree of Doctor of Engineering from the Carnegie Institute of Technology, Pittsburgh. 1919.

(35) **Westergaard, H. M.** Buckling of Elastic Structures. *Transactions,* Am. Soc. C. E., Vol. LXXXV (1922), p. 576.

(36) **Engineering-News Record,** 1920 and 1921.

THE HYDRAULIC DESIGN OF THE SHAFT SPILLWAY FOR THE DAVIS BRIDGE DAM, AND HYDRAULIC TESTS ON WORKING MODELS

Discussion*

BY MESSRS. E. W. LANE and I. GUTMANN

E. W. LANE,† ASSOC. M. AM. SOC. C. E. (by letter).‡—The author deserves the thanks of the profession for bringing to its attention the shaft spillway, which promises to be a useful device under certain conditions.

There are, in general, four phases which may be considered in discussing the shaft spillway, as expressed in the following questions: (1) What are the advantages and disadvantages of the shaft spillway? (2) What are the advantages and disadvantages of the particular shape used at Davis Bridge? (3) Could any other shape have been used to greater advantage? (4) Is the author's analysis, as developed mathematically and by experiment, correct in all particulars?

The spillway of an earth dam, especially one of the magnitude and location of that at Davis Bridge, is of vital importance, and a comparatively untried type should not be used without a thorough study, not only of the theory, but also of the possible, even though improbable, things which may happen to it. Taking this viewpoint, what are the advantages and disadvantages of the shaft spillway? The greatest advantage—in fact, the only one which seems prominent—is its low cost under such conditions as existed at Davis Bridge.

The danger of clogging from trees and drift brought down by a flood, especially at the bend, is probably the greatest disadvantage of the shaft spillway, and would seem to limit its use to large discharges. For example, recently in designing a spillway for a small earth dam, the writer found that the shaft type possessed great economy although the danger of clogging precluded it. The bend should be of sufficient radius to pass long logs without wedging; the combination of diameter of shaft and radius of bend of the Taf Fechan Spillway mentioned by Mr. Gourley‖ is near the lower limit of safety in this regard.

The probability of clogging is considerably greater than in the case of the conduit of a retarding basin, for the latter soon becomes submerged, and the drift is not drawn in; furthermore, the retarding basin conduit is usually straight. Yet, this contingency controlled, to a large extent, the selection of

* Discussion of the paper by Ford Kurtz, M. Am. Soc. C. E., continued from August, 1924, *Proceedings*.

† Asst. Engr., The Dayton Morgan Eng. Co., Pueblo, Colo.

‡ Received by the Secretary, June 27, 1924.

‖ *Proceedings*, Am. Soc. C. E., April, 1924, p. 562.

the size of the outlet conduits of three of the five basins of the Miami Conservancy District. The smallest discharge of any of these conduits, when the basin is filled to spillway level, is 8 800 sec-ft.

In cold climates there would be a possibility of such a spillway filling with ice. At Davis Bridge, as this formation would be ring-shaped, it would tend to hold itself in place by arch action. Furthermore, the air currents which would naturally form in the chimney-like tunnel would increase the tendency to freeze.

The disadvantage of having what is practically a maximum discharge has already been noted.* When the shaft spillway is flooded out, the discharge increases very little with an increase of head, probably even less in the actual spillway than is indicated by Fig. 25,† for the vacuum in the shaft would tend to exceed 33 ft., and as it cannot, the discharge would be somewhat reduced from that indicated by the model. What would actually take place under these conditions is difficult to visualize, but it is likely that some vibration would be set up; because of the great mass of rock surrounding the tunnel, this would do little damage except possibly to the tunnel lining.

The disadvantage of having a maximum discharge could be avoided if the shaft were designed so that this value would not be reached until the water in the basin rose to, or above, the elevation of the top of the earth section; but if sufficiently conservative predictions were used in determining the size of the maximum flood, the disadvantage of a limiting discharge would be more apparent than real. Every form of spillway for an earth dam has a limiting discharge, which is the capacity of the spillway when the water stands at the top of the earth dam.

Except for the vibrations that it might produce, the air drawn in to the spillway would be no disadvantage until it reduced the discharge capacity of the shaft to such an extent that the water backed up considerably above the level of the spillway crest. It is doubtful whether this would ever occur, for with the water backed up in this manner, the conditions causing the air to be drawn in would be largely removed, thus relieving the situation before it became critical, and, therefore, such a backing up would not result.

The foregoing discussion applies to shaft spillways in general; in addition, the form described in the paper seems to have other disadvantages which might not exist in all cases. The low head used on the spillway greatly increases the probability of clogging by drift, on account of the relatively low velocities of flow over the crest. The wide flat crest has a similar effect. If a tree with roots and branches reaches the crest of a high ogee dam and the water is deep enough, the root end is carried over and, after passing the crest, falls and raises the other end out of the water sufficiently to enable it to pass over also. With a crest width of 38 ft., and depths of water of 5 to 7 ft., at velocities of only 12 to 15 ft. per sec., such a possibility is doubtful.

This tendency of drift to catch on the crest is further augmented by the piers, and particularly by the bridge resting on their tops. Judging from the size of the model, there is a clear space of only 12 ft. between the weir crest

* *Proceedings*, Am. Soc. C. E., May, 1924, p. 673.

† *Loc. cit.*, December, 1923, p. 2035.

and the bridge, which would be inadequate to pass trees. The low velocities and shallow depths of the approach channel add another hazard.

It might be argued that in lakes such as that formed by the Davis Bridge Dam, floating drift could be easily kept from the spillway. This no doubt is true in ordinary floods, but it will never be possible to determine the adequacy of the equipment for large floods until they come, and then it will be too late to remedy any defect. Even a watchman permanently stationed at the dam might have his communications cut off by a large flood, and be unable to summon aid. At the Austin, Tex., dam, an immense raft has formed, covering the entire river for some distance above the dam with drift that has been prevented from passing over because of the gates and piers on the crest.

Some abrasion may occur at the bend in the shaft, due to the impact of ice and drift, as a great part of the material passing over the 500-ft. crest would strike on a width of about 30 ft. The water will be clear, however, and little damage is likely to result. At low discharges, a hydraulic jump will be formed in the conduit where the water moving at high velocity at the outside of the bend, is slowed down by the friction of the horizontal section of the tunnel. As the position of this jump will probably change with the discharge, and only clear water is to be expected, it is not likely to do appreciable damage.

Considerable value would have been added to the paper if the author had given the reasons for the selection of the amount to be used as the maximum head on the spillway, as this enters so vitally into the design. The low head necessitates a wide crest; but there is no apparent reason why a greater head could not have been used, making possible a smaller diameter, fewer gates, and less rock excavation. By using a well shaped ogee spillway, with a head of about 9.5 ft., instead of 7 ft., the length of crest could have been decreased to about half that used, and the danger of drift catching on the crest greatly lessened. This would have reduced the quantity of rock excavation in the approach channels, and as a concrete wall was required under part of the spillway lip in the form adopted, probably no great increase in the quantity of masonry would have been necessary. Such a spillway would have kept the water in contact with the spillway face, and have prevented a vacuum beneath the jet. Although it might not have been the cheapest form for this location, it would be well worth investigating in others.

Could some other form of shaft spillway have been used to greater advantage? It would be interesting to know whether any other shapes were investigated. At first thought, it would seem that a semi-circular, or a fan-shaped, design might decrease the cost, but a closer inspection shows that this is not necessarily so. With certain topographic conditions, considerable saving might result if these shapes could be used. They would have to be carefully studied experimentally, however, to guard against unexpected disadvantages from the hydraulic standpoint.

The writer believes that such experimental investigations generally pay big dividends, and should be used more frequently than they are. The initial expense looks large, and it is difficult to induce a client, who, without question,

will invest large sums in an enterprise with only 10 or 15% return, to invest a small sum in such an investigation, which will almost certainly yield this return, and may yield several hundred or a thousand per cent. Perhaps the most striking example of the solution of a difficult hydraulic problem by experiments on models was the development of the hydraulic jump pool for the outlets of the retarding basins of the Miami Conservancy District.* In this case not only the shape used, but the method itself, was determined as the result of the experimental work. Although the expense was considerable, it was amply justified by the saving in cost of construction.

Little fault can be found with the author's analysis of the hydraulics of the form of spillway used. The principal objection is to the neglect of the acceleration which takes place in a stream of water as it approaches a free drop, such as that which occurs at the beginning of the parabolic curved section. The hydraulics of this acceleration has been calculated by S. M. Woodward, M. Am. Soc. C. E.,† for a level bottom above the free discharge. Under these conditions, it was shown that the stream approaching the drop must flow at greater than the critical velocity, and this no doubt would also be true for the case of the spillway. If the up-stream corner of a flat-crested weir is rounded with a sufficiently large radius the critical velocity would be reached at the up-stream edge of the flat part of the crest, and the velocities down stream from this would be greater than the critical value, although for a very wide crest, friction might cause the point of critical flow to be farther down stream. If the radius of curvature of the up-stream corner is small, the point of critical flow would occur near the lower end of what the author calls the "normal acceleration section", and below this, subject to the friction effect already noted, the flow would be at a velocity greater than the critical, and, therefore, greater than that assumed by the author in computing the parabolic curve. Although the piezometric readings do not indicate a partial vacuum beneath the upper part of the nappe on the model it should be remembered that the flow for the conditions assumed in the design was only 3 in., and surface tension and frictional effects undoubtedly caused greater relative losses than would occur in a full-sized spillway.

Practically, however, it is probable that the parabolic curve is unnecessary, and that a somewhat sharper curve might be used with resulting economy. The objection to the vacuum beneath the nappe on an overflow dam is that it tends to overturn the dam, but this is not likely to be true of a shaft spillway, as the mass and arch action of the weir section would be ample to resist it. With gates on such a spillway, some of which were open, the nappe might tend to jump clear of the masonry surface, and possibly induce some vibration, but this would not be objectionable, as the spillway could hardly fail by sliding.

Summarizing, it is the writer's opinion that the shaft spillway is applicable only to large discharges; that the broad crest should be avoided by using

* "The Hydraulic Jump as a Means of Dissipating Energy," by Ross M. Riegel, M. Am. Soc. C. E., and John C. Bebee, Assoc. M. Am. Soc. C. E., Miami Conservancy District, Technical Reports, Pt. III.

† "Hydraulics of the Miami Flood Control Project," Miami Conservancy District, Technical Reports, Pt. VII, p. 136.

larger maximum heads, reducing the tendency to clogging, and probably the cost; that the disadvantage of having a limiting discharge is of no importance if sufficiently conservative values of run-off are assumed; and that the entraining of air is not likely materially to reduce the discharge capacity.

I. GUTMANN,* ASSOC. M. AM. SOC. C. E. (by letter).†—The writer wishes to adduce some European precedents in shaft spillway design.

The Krauserbauden masonry dam, in what is now Czechoslovakia, is provided with an auxiliary circular shaft spillway discharging into a tunnel which, in turn, empties into a pool formed by a weir built about 100 yd. down stream from the toe of the dam. The spillway shaft is about 16½ ft. in diameter and has a capacity of 3 500 cu. ft. per sec. for a weir head of 3.5 ft. Unlike the Davis Bridge Spillway, it has a distinct spill-weir rising 2 to 3 ft. above the ground. The spill-weir has a rounded crest and supports a vertical cylindrical screen about 10 ft. high. A circular footbridge surmounts the screen.

The masonry dam at Königreichwalde, also in what is now Czechoslovakia, is provided with two circular shaft spillways, one at either abutment, with tunnels leading into a water-cushion at the toe of the dam. The two spillways are not equal in capacity, one carrying about 4 000 cu. ft. per sec. and the other about 2 500 cu. ft. per sec. In their design, they are quite similar to the Krauserbauden shaft spillway.

Both the Krauserbauden and the Königreichwalde dams were built shortly before the beginning of the World War, by a Commission of the Kingdom of Bohemia ("Landeskommission für Flussregulierungen im Königreiche Böhmen"), appointed to form detention reservoirs for flood protection.

A multiple spillway, consisting of three shafts, 8.4 ft. in diameter and more than 60 ft. in depth, was designed by Professor L. V. Rossi, of Padua, for the Bassano Power Canal in Northern Italy.‡ The shafts are in a row, parallel to the axis of the canal; their upper ends are round-crested circular weirs rising 1.73 ft. above the ground, each weir having a perimeter of 79 ft., so that with a head of 2.46 ft., the capacity is expected to reach about 1 138 cu. ft. per sec. The three weirs are separated by only 1 m. in the clear from wall to wall. The canal and shafts are excavated in rock and lined. Ten grizzlies, 2.5 m. apart, are built in the vertical section of each shaft, in order to break up the force of the falling water.

The San Dalmazzo masonry dam in Northern Italy is provided with a unique spillway of ingenious design.‡ It is a multiple rectangular shaft spillway discharging into an open waste channel directly beneath. The dam, which is 190 ft. high, is situated in a narrow canyon 4 500 ft. above sea level, and controls a water-shed of 19.4 sq. miles, with a probable flood run-off of about 4 000 cu. ft. per sec., or 206 cu. ft. per sec. per sq. mile. A spillway length of 656 ft. was desired, but only a small, rocky plateau, 130 by 60 ft., was available as a site. The problem was solved by Fraccaroli and Piccinini, engineers

* New York, N. Y.

† Received by the Secretary, July 12, 1924.

‡ For details of the spillways of the Bassano Power Canal and the San Dalmazzo Dam see *Giornale del Genio Civile*, July 31, 1917, pp. 341–343, and Pl. IX.

of Societa Negri, in the following original manner: The plateau was excavated to a depth of from 6.6 to 11.5 ft. and a width of from 16 to 23 ft., to serve as the inlet to the wasteway. This inlet was bridged by a series of eight parallel concrete flumes resting partly on rock and partly on the walls of the inlet, with clearances of 6.9 ft. between flumes. The waste water enters the flumes from both ends and spills over their sides and through the clearances into the wasteway beneath. The first two flumes, beginning up stream, are 24 by 6.5 by 3.3 ft. deep; the remaining six are about 41 by 7 by 3.3 ft. deep, and the total structure is about 125 ft. long by 43 ft. wide. The total water-spilling perimeter is about 718 ft., that is, greatly in excess of the 656 ft. originally desired. In the winter of 1916-17, this spillway operated successfully with a head of about 1.7 ft. The stream carried ice floes from 1 to 1½ ft. thick at the time.

The author may be underestimating the reduction in capacity resulting from entrained air. Observations by the U. S. Reclamation Service on concrete chutes of the Boise Project in Idaho showed that at velocities of 15 to 30 ft. per sec., and depths of 3 to 4 in., the stream may entrain enough air to swell its original volume by from 30 to 50 per cent. The volume of air so absorbed seems to vary directly with the velocity and inversely with the depth of the stream. At least, the relation of volume of air entrained to velocity appears to vary asymptotically, the volume of absorbed air becoming noticeable in comparatively long runs at velocities in the vicinity of, say, 10 ft. per sec. This condition, therefore, can hardly be foretold from experiments on small scale models in which only low velocities are developed.

This brings out a limitation of the method of testing by models which may be eminently efficient in the study of straight-line and of curvilinear variations of a non-asymptotic nature, but will seldom prove applicable to the study of asymptotic variation.

THE SECONDARY EFFECT OF CERTAIN IMPORTANT RIVER BRIDGES ON LOCAL TRANSIT CONDITIONS

Discussion*

By Messrs. Herbert C. Keith and Ole Singstad

Herbert C. Keith,† M. Am. Soc. C. E.—The author has given a very good presentation of the problem. The speaker agrees with Mr. Fowler‡ as to the relative merits and proper use for bridges as compared with tunnels—that the tunnel is particularly suitable for public utility traffic, while the bridge is more appropriate under ordinary circumstances for private vehicular traffic.

There is one emphatic difference betweeen the New York-Brooklyn situation and that of Philadelphia-Camden which has not received the consideration deserved in the paper. At the time of the building of the Brooklyn Bridge—and of the later bridges between Manhattan and Brooklyn—Brooklyn was already a city of considerable size as compared with Manhattan, and entirely out of proportion to the present size of Camden relative to Philadelphia. Therefore, the increase of traffic resulting from the construction of the Delaware River Bridge should not be expected to parallel exactly that produced by the Brooklyn bridges. Brooklyn was—and still is—to a large extent a sleeping place for people whose business is in Manhattan: to some extent, Camden is similarly situated with relation to Philadelphia; but the fact that Brooklyn was already a large city when its bridges were built gave it less opportunity for expansion, and, therefore, gave less possibility for transit development as a result of building these bridges than may be expected in the case of Camden and Philadelphia.

The benefit to be derived from a bridge is dependent on the service that it gives to the public; and, consequently, the development of traffic across the bridge is dependent on the improvement of that service. This is ample explanation for the scant increase in traffic on the Brooklyn Bridge on the introduction of the shuttle service. If it is necessary to start at one edge of the river and stop at the other, and to take independent means of conveyance at either terminus, it breaks up the traffic too much for profitable use by the public.

With regard to the use to be made by public utility companies of the Camden-Philadelphia bridge, it seems self-evident that to obtain the greatest benefit for the public the Public Service Corporation of New Jersey, which

* Discussion of the paper by John A. Miller, Jr., Assoc. M. Am. Soc. C. E., continued from August, 1924, *Proceedings*.

† Cons. Engr., New York, N. Y.

‡ *Proceedings*, Am. Soc. C. E., August, 1924, p. 863.

feeds from and distributes to the New Jersey side, should be the company to bring passengers into and take them from the great city, rather than the Philadelphia transit lines that should carry their traffic just outside the city for distribution over the larger area. The latter would be of comparatively slight benefit to the public. The Philadelphia company probably would not receive enough benefit to warrant the extra expense, whereas the New Jersey companies would derive a material benefit, because they could carry their passengers to and take them from the heart of the city—a great boon to their patrons. It is fair to predict that such benefit would increase through a long period of years, and not simply be a temporary improvement followed by only slight progress. Such traffic controlled by the New Jersey corporation, if properly managed and supervised, will grow increasingly as Camden and its vicinity develops.

This is another illustration of the old rule that as the population increases traffic will increase in proportion to the square of the population. The figures given by the author refer only to the population of Brooklyn: had they included the whole territory tributary to the Brooklyn Bridge instead of simply the Brooklyn end of it, probably the old rule would have found verification.

OLE SINGSTAD,* M. AM. SOC. C. E.—The speaker would first like to refer to some of the points brought out in the discussion, before presenting certain ideas regarding the paper itself.

He heartily agrees with Mr. Fowler† that, in the planning of the Delaware River Bridge, the vehicular traffic has undoubtedly been one of the determining factors, although this phase has not been discussed in the paper. He must disagree, however, on one or two other points. Mr. Fowler stated that tunnels are particularly well adapted for rapid transit purposes in a river crossing, but that there are physiological and psychological objections to tunnels for vehicular traffic. The speaker fully agrees with the first part of that statement but not with the second part.

The exhaustive investigations carried out by the New York State Bridge and Tunnel Commission and the New Jersey Interstate Bridge and Tunnel Commission in connection with the construction of the Hudson River Vehicular Tunnel, have demonstrated that vehicular tunnels can be adequately ventilated at a reasonable cost, and that there need be no physiological objection to the use of tunnels for vehicular traffic any more than to the use of streets with heavy motor traffic.

The psychological objections to vehicular tunnels are refuted by the experiences of the vehicular tunnels under the Thames in London, England, namely, the Blackwall Tunnel, which has been in successful operation for vehicular traffic for twenty-seven years, and the Rotherhithe Tunnel which has been in operation for sixteen years. These tunnels, 4 465 ft. and 4 930 ft. long between portals, respectively, are used by a mixed traffic consisting of horse-drawn,

* Engr. of Designs, New York and New Jersey State Bridge and Tunnel Comms., New York, N. Y.

† *Proceedings*, Am. Soc. C. E., August, 1924, p. 863.

gasoline-propelled, and coal-burning steam-driven vehicles. Until about two years ago, the only ventilation was by natural draft through four shafts and the portals. Then a small mechanical ventilation plant was installed in the Blackwall Tunnel. In the summer of 1923, the speaker passed through these tunnels several times and found no evidence of any hesitancy to use them, even with the rather inadequate ventilation, and the engineer of the London County Council in charge of the operation of the tunnels stated that at no time had there been any interruption in traffic for this cause.

The speaker also disagrees with another statement made by Mr. Fowler, namely, that all tracks should be removed from the East River bridges and all available space on these bridges utilized for vehicular traffic, while the tracks should be carried in tunnels under the river. Such an arrangement appears advisable on its face, but would not be feasible in practice for the simple reason that, if the bridges were to be operated to their full capacity under those conditions, the existing approaches and adjacent city streets, or any approaches which could be created at a reasonable cost, would be overcrowded, and the resulting congestion would preclude the use of the bridges to their full capacity.

The paper is devoted principally to the discussion of the transit conditions at Philadelphia, Pa., and Camden, N. J., in relation to the Delaware River Bridge now under construction. The author has discussed several ways of utilizing the tracks of the bridge for transit purposes, but, in the speaker's opinion, has found no satisfactory solution of the problem. This is in no sense a criticism of the paper. The reason for this failure is more fundamental; such a solution was precluded when the type and location of crossing were determined. It appears that in deciding on these features, the requirements of vehicular traffic were given prior consideration to the transit requirements.

The delivery district for passengers in Philadelphia may be defined as a narrow rectangle about $1\frac{3}{4}$ miles long, extending from the Delaware River to 20th Street with Market Street as the east and west axis, and $\frac{3}{8}$ mile wide extending from Arch Street, one block north of Market Street, to Walnut Street, two blocks south of Market Street. (See Fig. 7.)

It will be noted that the Philadelphia terminus of the bridge is entirely outside this district, being about $\frac{1}{4}$ mile north of its northern limit and near its eastern boundary. This means that under the best plan of transit operation suggested by the author, the "Camdenites" would not be delivered at or near their points of destination in Philadelphia. Further, the bridge terminus is not conveniently located on a transit line running longitudinally through the delivery district.

The author has stated that 77% of the entire passenger traffic between Philadelphia and Camden uses the Market Street Ferry. At the Camden terminus of this ferry are the Pennsylvania Railroad Station and the terminus of practically all the Camden surface lines. One-third of the passengers crossing on the Market Street Ferry from Philadelphia, that is, about one-fourth of the total traffic, are destined for the Pennsylvania Railroad Station, and a

large part of those remaining take the trolley at the same place. The Camden terminus of the bridge is more than ½ mile from this point (see Fig. 7).

Evidently, as far as passenger traffic is concerned, the bridge is not located so as to meet the requirements on either side of the river, doubtless because in planning the crossing, the engineers had to consider the requirements of the vehicular traffic, the passenger traffic, the congestion on the city streets, and real estate values. All these conditions could not be met satisfactorily with a bridge.

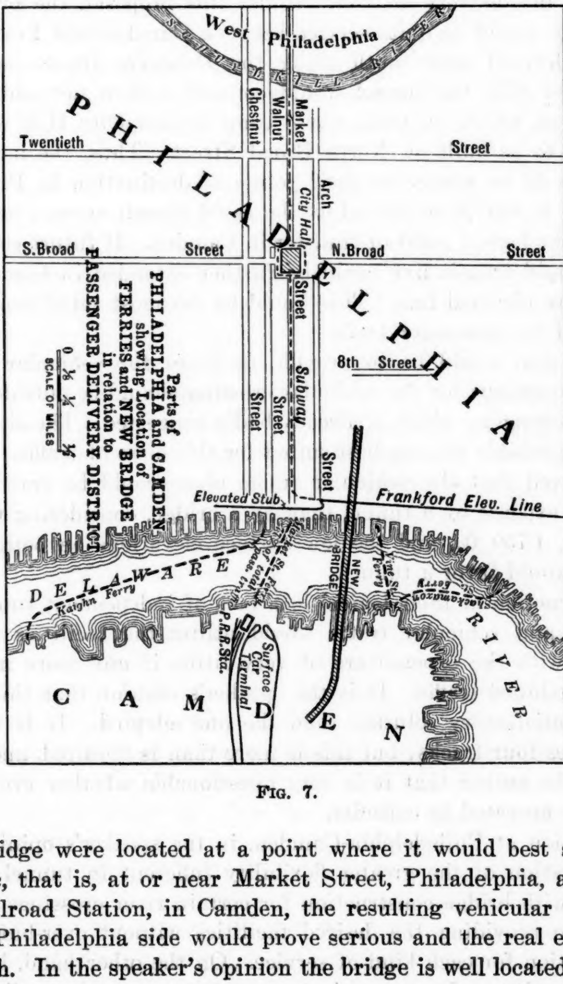


FIG. 7.

If the bridge were located at a point where it would best serve the passenger traffic, that is, at or near Market Street, Philadelphia, and the Pennsylvania Railroad Station, in Camden, the resulting vehicular traffic congestion on the Philadelphia side would prove serious and the real estate damages would be high. In the speaker's opinion the bridge is well located for vehicular traffic, but will offer little advantage over the present facilities for passenger transportation.

In 1918 and 1919, the speaker had occasion to study carefully the problem of a crossing between Philadelphia and Camden, both for passenger and

vehicular traffic. The traffic counts taken at that time showed that two rapid transit tracks would take care of the passenger traffic for a number of years, and that roadway facilities for two lines of vehicles in each direction would take care of the vehicular traffic for a corresponding period.

It appeared to the speaker then (and he is still of the opinion) that the most practicable crossing for passengers would be a rapid transit tunnel from a terminal station on the Camden side near the Pennsylvania Railroad Station and the terminus of the Camden trolley lines, under the Delaware River and connecting with a rapid transit line in Philadelphia running east and west through the delivery district. Under this proposal, the present Market Street Subway would split its service between Camden and Frankford, abandoning the elevated spur south along the Delaware River water-front; or probably better still, the tunnel would connect with a new subway through Chestnut Street, which, in turn, would connect near City Hall with the four-track subway to be built on North Broad Street. Thus, the majority of the passengers would be carried to their points of destination in Philadelphia or could proceed to any point served by the rapid transit system, and would also be carried to the logical point of transfer in Camden. If future conditions warranted, the rapid transit line could be further extended in Camden either as a surface or an elevated line. This would be the most satisfactory method of taking care of the passenger traffic.

The next step would be to provide facilities for vehicular traffic. The Philadelphia terminus for the vehicular crossing should be outside the delivery district for passengers, which is already badly congested. The site selected for the bridge is probably the one best suited for this class of traffic.

It is believed that the vehicular traffic alone could be cared for at considerably less expense by a tunnel than by a bridge, considering the long span of the bridge, 1 750 ft., and the low surface elevations on both sides of the river, which would favor a tunnel.

The construction at their logical locations of independent tunnels for both rapid transit and vehicular traffic would utilize the facilities to better advantage and with the expenditure of very little, if any more money than a bridge for combined traffic. It is the speaker's opinion that this would be a much more satisfactory solution than the one adopted. It is true that the bridge provides four tracks; but this is more than is required, and the speaker agrees with the author that it is very questionable whether even two of the tracks will be operated to capacity.

The situation at Philadelphia-Camden, in the speaker's opinion, is an excellent illustration of the greater flexibility inherent in tunnel construction as compared with bridge construction for certain river crossings. The tunnel lends itself to providing the desired facilities without over-building and at the ideal location for each kind of service. On the other hand, in the case of a bridge the tendency, for economic reasons, is to concentrate all traffic at one point, thereby causing congestion at the approaches and sacrificing the most desirable location for one class of traffic or the other.

INCREASING THE CAPACITY OF EXISTING STREETS

Discussion*

By William T. Lyle, Assoc. M. Am. Soc. C. E.

William T. Lyle,† Assoc. M. Am. Soc. C. E. (by letter).‡—The problem of increasing the capacity of streets is one of interest not only to great cities, but to small towns as well. In many of the latter serious congestion is now being experienced, especially in those through which pass great State and National highways.

In Southern towns the streets are quite generally narrow, from 25 to 30 ft. between curb lines, and sometimes even less. In Colonial days, these widths were ample and remained so until the advent of the automobile; in fact, no serious trouble was experienced until recently. Local motor cars have multiplied, but more important in some respects is the great surge of State and interstate traffic, creating a condition which the citizens in their own business interests and for their own comfort and protection cannot ignore.

The authorities of many Southern towns on the Lee, Bankhead, Dixie, and Southern National Highways have realized this and are now seeking relief in police regulation of traffic, restricted parking, and the entire elimination of parking. At first, parking on one side of the street is prohibited; then parking on both sides. Finally, the merchants come to a realization of the effects of these regulations on their business and clamor for the removal of the parking restrictions. Evidently, however, such a return is impossible unless a drastic procedure is adopted—either the widening of the street or the restriction of traffic to one direction. To those who seek a profit from the tourists, the latter solution is unsatisfactory.

In addition to the local interests, however, there are also great outside interests of State and National character. The State Highway authorities realize the throttling effect of towns through which the highways pass and, therefore, can be expected to co-operate with the local authorities in re-location work and bridge construction when the benefits belong more to the State than to the locality.

Specifically, the town of Lexington, Va., the resident population of which is 3 000, will be taken as an example. It is a university town, the seat of two institutions of learning and the burial place of Generals Lee and Jackson. It is often spoken of as the "Shrine of the South"; gauged by the volume of incoming and outgoing motor traffic, it is indeed becoming more and more so.

* Discussion of the paper by Arthur S. Tuttle, M. Am. Soc. C. E., continued from August, 1924, *Proceedings*.

† Prof. of Civ. Eng., Washington and Lee Univ., Lexington, Va.

‡ Received by the Secretary, June 23, 1924.

Until recently this traffic was inconsiderable; but it is now increasing rapidly due to the development of the Lee Highway through Virginia, now nearly completed. A heavy burden is thus thrown on streets which were intended to serve very different purposes. The State Highway Department cannot be expected to assume the expense of street widening; local interests must work out their own salvation. In the center of the town, the highway measures 30.6 ft. between curbs and 50.7 ft. between building lines.

A plan has been prepared to widen the street to the face of an important bank building which was built 6.15 ft. back of the building line, thus giving a total width of 56.85 ft. between building lines and 38.85 ft. between curbs, the removal of poles on both sides of the street providing an extra 2.1 ft. between curbs. Of the cost of this improvement, 60% should be assessed as benefits on abutting properties, the remainder being provided from the general taxes.

A second solution of the problem is to provide for double routing between points of junction, no street widening being necessary. This is possible by reason of the existence of two parallel streets through the town, 300 ft. apart, that meet at an acute angle at the northern end of the business district, one of them being the main street. At the southern end of the town, it would be necessary to create a similar junction by a re-location of one street and the removal of one or two houses. This solution presents less obstacles and could be effected at small expense. It would not prove satisfactory to the business interests of the town as they now exist, however, because it would eventually result in the transformation of the street that is now partly undeveloped and partly residential, to one lined with business houses catering to the needs of tourists.

In either case, however, for at least 0.8 mile, the approach to the business district would have to be widened, as the roadway is only 22 ft. between curbs and inadequate even at the present time with the Lee Highway still uncompleted. Traffic censuses show great overcrowding with the prospect of a continuing increase. To provide for it, the street should be widened to 26 ft. between curbs, with a re-location of the highway in the northern end of the town to eliminate dangerous curves. A new bridge with approaches will also become necessary, a part of the cost of which should be borne by the State.

Under the first plan, space is provided in the heart of the town for parking cars from both directions in the middle of the street at an angle of 45° to the curb. This treatment is believed to be unusual, but has much to recommend it. With the second plan, by which double routing is proposed, ample parking space on one side of each street is provided. The roadway width of 26 ft., previously mentioned for the northern approach to the point of junction, will provide for parking space on one side of the street with ample clearances for cars traveling in opposite directions at fairly high speeds.

In order to save expense, setback lines on the main street have been considered; eventually, they would achieve the desired result of street widening, as new buildings could not be constructed in front of these lines, nor could reconstruction be carried beyond them. This plan evidently would involve

great delay in effecting the result desired and is not favored for that reason. If a plan for immediate widening is adopted, the arcading of certain buildings might seem desirable owing to their historical value.

Lastly, there is the troublesome problem of assessments for benefits. In Virginia, assessments of this kind can probably not be levied under the amended Constitution of 1902, which reads:

"No city or town shall impose any tax or assessment upon abutting land owners for street or other public local improvements, except for the making and improving the walkways upon then existing streets, and improving and paving then existing alleys, and for either the construction, or for the use of sewers; and the same when imposed, shall not be in excess of the peculiar benefits resulting therefrom to such abutting land owners."

Evidently, this section is in need of further amendment.

IMHOFF TANKS—REASONS FOR DIFFERENCES IN BEHAVIOR

Discussion*

By Morris M. Cohn, Esq.

Morris M. Cohn,† Esq. (by letter).‡—The operation of the Imhoff tanks at Schenectady always has been and, no doubt, always will be the cause of much trouble. Only by the application of much labor and time can they be made to function properly and as yet no means have been discovered that will produce normal operation.

It appears from experience, that, if left without attention, the tanks will not foam. The scum blanket will increase in depth and dry out, producing an apparently foam-proof layer through which the gases cannot escape. Furthermore, any attempt to draw sludge will meet with failure. Hosing of the scum accumulation will drive down the mass and make possible the drawing of a mixture of sludge, scum, and water. This is proof that the sludge obtained cannot be truly normal. A well-behaved Imhoff tank will always have the oldest sludge immediately tributary to the sludge pipe; but the constant agitation of the Schenectady sludge gives a mixture of old sludge, moderately fresh sludge, and undigested particles just deposited. Such a mixture can never attain the condition that typical sludge should have when drawn for dewatering. It is only at the height of the warm season, when digestion is rapid, that this mixture reaches a reasonably good condition. This heterogeneity accounts for the fact that analyses show only a slight variation in the composition of the sludge from the first drawing of the season until the final drawing in the fall.

Thus, it is the formation of scum rather than of sludge that can be held accountable for the tank difficulties at Schenectady. Two conditions may cause the rise of the solids deposited in the sludge compartment of the tanks: (1) the solids may be lighter than the sewage, due either to their own character or to the effect of some outside agency; and (2) the entrained gases may be held so tenaciously that they do not escape when the solids rise in the gas vents, thereby giving the mass a specific gravity less than that of the liquid. It is impossible to ascertain which of these factors is the cause of the difficulty, or whether both causes contribute to the general condition of the tanks; it suffices that the net result is the same whatever the specific cause.

---

* Discussion of the paper by Harrison P. Eddy, M. Am. Soc. C. E., continued from August, 1924, *Proceedings*.

† Supt. of Bureau of Sewage Disposal, Dept. of Public Works, Schenectady, N. Y.

‡ Received by the Secretary, July 17, 1924.

During the summer of 1922 the majority of the sludge samples drawn for test had specific gravities less than water. Furthermore, a thorough agitation of the samples failed to liberate the gases sufficiently to increase materially the density. Mr. Eddy suggested that the sludge be placed in a freezing solution and then agitated to free the gases. Only after such treatment was it possible to obtain gravities greater than 1.0. This condition indicates that the sludge in the hoppers is composed of particles of low specific gravity easily buoyed up by relatively small quantities of entrained gases.

There has always been a tendency for the fresh solid particles deposited in the hoppers to rise to the surface of the gas vents. On the resumption of operations in 1921, following a period of idleness for repairs, the tanks started in a clean condition. Despite the fact that operation was commenced the first week in November, when the temperature was quite low, the first few days of operation produced a covering of fresh solids in the vents of the tanks. This rise of the solids may be attributed in part to the formation of gas, but also in part to the inherent lightness of the solid particles themselves, because of their own characteristics or because of some outside agencies.

It is not easy to ascertain the cause of the solids and sludge rising in the vents. One thing is certain, namely, that at times the city sewage contains a great deal of grease and oil. During one of the worst days there has been removed as many as sixty wheel-barrows of black oil and grease from the inlet to the tanks. As further proof that the sewage flow contains a great quantity of oil and grease it can be stated that the exposed concrete about the tanks is stained by grease and that a thick gelatinous grease layer is found in the effluent channels and dosing tanks. Grease balls are a great cause of nozzle clogging on the trickling filters, and the spindles of the Taylor nozzles are always coated with a heavy grease layer. The bottles in which the daily sewage samples are collected and in which the composite weekly samples are made up, are oily in appearance. Further, the sludge after de-watering has an oily smell. Although most sewages contain a certain quantity of oils and grease, still it may be true that the raw material at Schenectady contains the same elements in an excessive degree. This grease may produce a tenacious, waterproof coating to each particle of sludge and make it impossible for the entrained gases to break free, allowing the mass to settle. These small particles may be likened to gas-filled balloons that are set free and fly to a great altitude. As the balloons rise, the gas tends to expand, due to the decrease in atmospheric pressure. If the material of which the balloons are made is weak, this expansive force will burst them and they will fall to earth, but when the enclosing shells are so tough that the internal pressure will not rupture them, the balloons will stay suspended as do the particles of Schenectady sludge. The action of the sludge samples of 1922, referred to previously, proves either that the gases were held so tenaciously that beating would not release them, or that the gas formation was so rapid as to offset the rate of release. On the other hand, it may be that the gases are under so low a pressure, due to the shallowness of the Schenectady tanks, that they would not have sufficient expansive force to break free from a sludge that offered even less resistance than the

Schenectady sludge. At any rate the grease theory appears to warrant consideration.

In reference to the quantity of soap curd produced in this city, it will be interesting to quote from an investigation of the hardness of Schenectady tap water carried out by the writer in 1921 for the purpose of determining the feasibility of installing a softening plant. By a series of laboratory experiments the waste ratios for representative soaps were determined. This waste ratio is the ratio of the quantity of soap required to soften the water to the total quantity used. It is sufficient to state that the average ratio for all soaps was found to be about 6% with a water of the hardness of Schenectady's supply. It can readily be seen that the soap curd produced by 6% of all the soap used by the population tributary to the plant will be very large in the aggregate. On the basis of the hardness of the water in parts per million, this quantity per 1 000 000 gal. of sewage at Schenectady would be twice as great as at Rochester and thirteen times as great as at Fitchburg. This curd, no doubt, acts as a coating for the sludge particles and produces the same effect as oils and greases. As Mr. Eddy states, it surely adds to the load of suspended solids on the units. It certainly appears that the great quantities of soap curd must be the cause of some of the difficulties at Schenectady.

It is of interest to ascertain whether or not there is any inhibitory action produced on the sludge by the materials already in the sewage, or introduced into it during the ordinary course of treatment. The sewage contains a great deal of mash and grain which comes, no doubt, from the illicit manufacture of liquor. Many sewers in the city have been found to be completely choked with this material. It seems that all this matter finds its way into the gas vents and cannot be driven down for any length of time. One-half hour after the scum blanket in a gas vent has been driven down by the use of a 1¼-in. stream of water, there will appear a layer of at least ½ in. of swollen and puffed grain on the surface. Below this, the scum builds up, but never yet has the grain failed to rise before the sludge. A distinct sour odor was noticed during the spring of 1924 when the vents were first hosed. There is no question but that true fermentation is taking place in the sludge hoppers, and it may be that the carbon dioxide emitted retards the digestion of the sludge.

The effect of the water introduced into the vents during the process of hosing is debatable. It might be expected that the dissolved oxygen would tend to stimulate the kind of biological action which produces carbon dioxide and that this would affect adversely the digestion process. An experiment in which sewage was used to drive down the scum on one tank and water to perform the same duties on the other tanks under the same conditions disclosed the fact that all the tanks acted similarly in regard to the formation of scum. It can be said, nevertheless, that the addition of water to the scum blanket does seem to produce the phenomenon of foaming. At Schenectady, this seems to be due to the softening of the scum layer rather than to any gas action. It appears that the dried scum retards the passage of gases and exerts a downward pressure that offsets the tendency of the vents to foam.

Under this condition a great deal of sludge finds its way into the sedimentation chambers through the slots, the material following the path of least resistance.

A few other phenomena have been noticed by the writer, that seem to bear out the fact that the Schenectady sludge has an inherent tendency to rise. When the tanks were cleaned during the spring and early summer of 1923, a small quantity of sludge was left in the bottom of the hoppers for seeding purposes. It was noticed that in a few hours the water had settled out of this residue leaving the sludge as a heavy scum on the surface. This was so marked that the material was stirred well before samples were taken for analysis. There is another point that may be of interest: When during the summer of 1923 sludge was drawn from the tanks immediately after hosing, in many instances, there would be a good flow of moderately heavy sludge for a period, followed by a black septic liquid. No more sludge could be drawn despite the fact that the gas vents were clear. After a few hours, scum would again form and by the end of 24 hours a heavy deposit would have gathered in the vents. The question arises as to where the solids were at the time of drawing. It might be expected that the driving down of about 3 or 4 ft. of scum would produce a sludge deposit tributary to the sludge pipe that could be drawn at will. This was not the case, and it certainly looks as if the buoyancy of the scum was not destroyed by the action of the 1¼-in. stream of water under 60 lb. pressure. Most of the scum must have been suspended somewhere between the surface of the vents and the bottom of the tanks.

An investigation was begun early in 1924 to determine the reaction or hydrogen-ion concentration of the sewage liquid drained from the material in the sludge and scum compartments, and of tank effluent. Table 19 gives the results of the pH determinations to date.

TABLE 19.—Hydrogen-ion Contentration at Schenectady, N. Y., 1924.

Date.	Raw sewage.	Tank effluent.	Drain liquor from sludge.
April 11, 1924	7.1	6.9	....
April 17, 1924	6.8	6.6	7.3
April 26, 1924	6.8	6.6	7.3
May 2, 1924	....	....	....
May 9, 1924	7.2	6.9	....
May 16, 1924	7.3	6.8	....
May 21, 1924	....	....	7.9
May 30, 1924	8.0	6.8	....
June 3. 1924	....	....	7.8

There appears to be a distinct increase in the acidity of the sewage in its course through the tanks, and the sludge liquor appears less acid than either of the other samples. If the pH values are indicative of the bio-chemical changes occurring in the tanks, it will be of interest to watch the effect of the later warm weather on them.

The question of temperature effects on the operation of the Schenectady tanks should be answered by the record of the temperature of the sludge in the digestion compartments now being obtained by the use of a compensated re-

cording thermometer. The average monthly temperatures of air, crude sewage, tank effluent and sludge compartment contents since the installation of the gauge is shown in Table 20.

TABLE 20.—TEMPERATURE OF SCHENECTADY SEWAGE, IN DEGREES FAHRENHEIT.

Month.	Air.	Crude sewage.	Tank effluent.	Sludge compartment.
1923				
December	36.6	52.5	51.9	50.0
1924				
January	27.6	51.1	50.3	50.0
February	22.2	49.1	48.3	48.0
March	36.8	48.9	47.9	48.0
April	46.8	48.9	48.5	48.5
May	55.8	52.0	51.5	53.0

The first signs of any distinct gas formation in the vents were detected in May, during which month the highest temperature recorded for the material in the sludge compartments was 57° Fahr. It is certain that little digestion can go on at temperatures lower than those experienced in May. In June, with an air temperature of 80° Fahr., the temperature in the sludge compartment reached 61° Fahr. Evidently, the period of active digestion of sludge will be short, and the great mass of the deposits will produce much gas. This large-scale formation of gas may be the cause of the foaming tendency that is already becoming evident. The continuation of the temperature record should throw some light on the effect of temperature on the digestion of sludge in the Imhoff tanks at Schenectady.

MEMOIRS OF DECEASED MEMBERS

NOTE.—Memoirs will be reproduced in the volumes of *Transactions*. Any information which will amplify the records as here printed, or correct any errors, should be forwarded to the Secretary prior to the final publication.

HENRY MARISON BYLLESBY, M. Am. Soc. C. E.*

DIED MAY 1, 1924.

Henry Marison Byllesby, the son of the Rev. De Witt Clinton Byllesby and Sarah (Mathews) Byllesby, was born in Pittsburgh, Pa., on February 16, 1859. He was of English descent, his American lineage on his mother's side dating to 1620 and on his father's side to 1789.

Mr. Byllesby took a preparatory course in the Western University of Pennsylvania, from 1871 to 1873, and a course in Mechanical Engineering at Lehigh University from 1873 to 1877, leaving at the end of his Junior year to engage in practical work with the Weston Shops at Newark, N. J.

In November, 1879, he was employed as a Draftsman by Robert Wetherill and Company of Chester, Pa., leaving that firm on June 1, 1881, to enter the employ of the Edison Company, where he was engaged as a Draftsman on the plans of the First District Pearl Street Station of the original Edison Electric Illuminating Company of New York. From November, 1882, to February 1, 1885, he was on the general Engineering Staff of the Edison Company, engaged in installation work in the United States and Canada. Thereafter until November 1, 1885, Mr. Byllesby represented the firm of Robert Wetherill and Company as Eastern Manager. He was then engaged, with several others whose names have since become familiar in the electrical industry, by the late George Westinghouse, M. Am. Soc. C. E., who at that time was conducting his electrical experiments, as a personal venture, in the shops of the Union Switch and Signal Company, which he then controlled. Soon after the Westinghouse Electric Company was organized, with Mr. Byllesby as Vice-President and General Manager. The rapid development of the electrical industry and the prominent place assumed in it by the Westinghouse Company, which soon absorbed the Union Switch and Signal Company, gave him ample opportunity to exercise his talents, and resulted in his taking out, either personally or as joint inventor, numerous patents relating to electrical lighting apparatus, and also in his representing his Company in Europe from May 11, 1889, to September 1, 1891.

In December, 1891, he resigned and, on February 1, 1892, was engaged by the Thomson-Houston Electric Company, becoming, on April 1, President of the Northwest Thomson-Houston Company, a subsidiary organization, with headquarters at St. Paul, Minn. He remained there until the spring of 1895, the Company at that time being absorbed by the General Electric Company. For the next seven years Mr. Byllesby's energies were occupied in developing various important hydro-electric and other properties throughout the Northwest.

* Memoir prepared by Bion J. Arnold, M. Am. Soc. C. E.

On January 1, 1902, he moved to Chicago, Ill., and organized the H. M. Byllesby and Company, Incorporated, of which he was President. Originally established as an engineering firm, the Company soon became interested in the financing, design, construction, operation, and management of electric and gas properties. During recent years, it has devoted itself to the underwriting and distribution of investment securities, the Byllesby Engineering and Management Corporation having taken over the operating, managing, and engineering work. Mr. Byllesby was either President or Director of many of the large utility companies of the United States, and was the pioneer in the establishment of the customer-ownership policy which is now so popular.

Notwithstanding his numerous affiliations, he found time to aid in civic work, and served with distinction as President of the Chicago Civic Federation. As Chairman of the Executive Committee of the Chicago Branch of the National Security League, he was largely responsible for inaugurating the rousing patriotic speaking campaign in Chicago, the West, and the Northwest, during the fall and winter of 1917.

Having offered his services at the entrance of this country into the World War, Mr. Byllesby was commissioned as Major in the Recruiting Division of the Signal Corps, U. S. Army, on November 15, 1917. Later, he went overseas, was promoted to the rank of Lieutenant Colonel, and served as the London representative of the American Expeditionary Forces, having charge of purchases in London and the Scandinavian countries. On December 19, 1918, he was honorably discharged and returned to his civilian duties. He was awarded the Distinguished Service Order by the British Government, and, on August 21, 1922, the Distinguished Service Medal of the United States, for "exceptionally meritorious and distinguished services." In 1923, he was commissioned a Colonel, Corps of Engineers, in the U. S. Officers' Reserve Corps.

In October, 1923, on account of serious heart trouble, Colonel Byllesby was compelled to drop all business cares, and he and Mrs. Byllesby spent the winter in California, returning to Chicago on April 17, 1924. He was in good spirits and confident that he would soon be able to resume his business duties. It was hoped that he was recovering when his death came unexpectedly on May 1, in the office of a dentist, where he had gone for a treatment, accompanied by his nurse.

Funeral services were conducted in St. Paul's Protestant Episcopal Church, Chicago, on May 3, at which the Right Rev. Herman R. Page, Bishop of Michigan, an old and intimate friend, and a former Rector of St. Paul's, spoke on the life and personality of Colonel Byllesby. Following the services the funeral party was taken by special train to Lake Geneva, Wis., where the burial rites were conducted with full military honors; an honor guard from Fort Sheridan, the cadets of the Northwestern Military Academy, and members of Byllesby Post of the American Legion, participated.

Colonel Byllesby was married to Margaret Stearns Baldwin, of Roselle, N. J., on June 15, 1882, who survives him. They resided in Chicago and Holly Bush House, Lake Geneva, Wis.

He was a member of the American Society of Mechanical Engineers, the Western Society of Engineers, the Public Policy Committee of the National Electric Light Association, and a Fellow of the American Institute of Electrical Engineers. He was also a member of the leading clubs of Chicago, New York, Minneapolis and St. Paul, Minn., Louisville, Ky., Portland, Ore., and San Francisco and San Diego, Calif.

Colonel Byllesby had great civic and National pride, and gave freely of his strength and means to all movements for education, social betterment, and the advancement of patriotism. In his death the country has lost an energetic and valuable citizen; the utility field, a liberal and progressive leader; the Engineering Profession, an able member; and those who knew him intimately, a generous and whole-souled friend.

Colonel Byllesby was elected a Member of the American Society of Civil Engineers on June 1, 1887.

ALBERT LADD COLBY, M. Am. Soc. C. E.*

Died April 30, 1924.

Albert Ladd Colby was born in New York, N. Y., on June 26, 1860, of John Ladd, M. D., and Mary Ann Colby. He was educated in the public schools of New York, at the College of the City of New York, and in 1881 received the degree of Ph. B. from the Columbia School of Mines. Mr. Colby continued his studies at Columbia, acting occasionally as Assistant to Professor C. F. Chandler, until 1883.

For the next three years he was Assistant Professor of Chemistry at Lehigh University; then in turn Head Chemist and Metallurgical Engineer of the Bethlehem Steel Company, at Bethlehem, Pa., occupying the latter position until 1903. As Head Chemist, Mr. Colby's active and inventive mind devised several improvements in apparatus for increasing the rapidity and accuracy of the analytical work; as Metallurgical Engineer, he was, from time to time, in charge of the Blast Furnace, Open-Hearth, Bessemer, and Puddling Departments; he also had entire charge of the inspection, assignment, and economic use of all metallurgical materials, and he early gave his attention to the formulation of specifications for such materials—an activity in which he long held a distinguished part in the United States, and as its representative in England and Europe.

Mr. Colby also served as Expert for the Bethlehem Steel Company in patent suits, again laying the foundation for valuable service later as a Consulting Engineer. In this latter capacity, he was excelled by few, if any, contemporaries in the iron and steel industry. His mind was ingenious, thorough, clear, and keen, both in the preparation of evidence and its presentation to the Court. He very quickly appraised the value of evidence, or of an answer to a question, and knew how to secure the information, either from the available literature of any language, or by special researches or plant studies. In

* Memoir prepared by Bradley Stoughton, Esq., Bethlehem, Pa.

the executive control of literary research for patent purposes, his long training, infinite capacity for digging through long series of articles in English, German, or French, and his clear formulation of the vital subject-matter for the guidance of assistants, made him invaluable.

Mr. Colby was also an adept in collecting information on special branches of his profession. For example, when the science of Metallography began to assume importance, he made the first really comprehensive bibliography covering the literature of all siderurgical countries. For the newly-formed International Nickel Company, he gathered all the information extant on the subject of nickel steel, both in the literature and in the minds of experts in America and Europe. Two books testify to the thoroughness of his work in this capacity, namely "American Standard Specifications for Steel", 1902, and "Reinforced Concrete in Europe", 1909, besides many published monographs.

On the completion of his work for the International Nickel Company, Mr. Colby practiced as Consulting Engineer, with an office in New York, and, later, in Bethlehem also. He practiced a great deal in Europe, representing American firms and inventors there, for which his familiarity with the French and German languages and his wide acquaintance among the prominent iron and steel men especially fitted him. Although he was connected with the Bethlehem Steel Company, from 1886, he was in demand by other companies as Consulting Engineer or expert in patent suits. In 1906, he began to specialize in by-product coke manufacture, and practiced in this industry for several years. He was representing an American firm abroad, when he was stricken with influenza in London, and died at Torquay, England, on April 30, 1924.

On June 20, 1894, Mr. Colby was married to Agnes Wilson Lee, of Lewistown, Pa., who, with two children, survives him.

Mr. Colby will be remembered particularly for his life-long work on the specifications of steel, for his contributions to the advance of metallurgical knowledge in his several compilations of specialized information, for his public work as a representative of the industry, and for his active and disinterested support of technical societies.

In 1900, he represented the Bethlehem Steel Company at the Paris Exposition, and served as Juror in Metallurgy and Official Representative of the American Association of Steel Manufacturers; he was also United States Representative at the International Congress for Testing Materials of Construction. He was Special Iron and Steel Commissioner in the Department of Mines and Metallurgy at the St. Louis Exposition, in 1904, and, during the World War, took an active part in the Standardization of Aircraft Steels. He was Secretary of the American Association of Steel Manufacturers, which, during his incumbency (1897 to 1905), made the first successful effort to standardize specifications for finished steel, and he served actively on Committees of the American Association for Testing Materials, and its international associate, in formulating steel specifications. Space does not permit the listing of all the committees on which he served in his various technical societies, which included the Iron and Steel Institute of Great Britain, the American Society of Mechanical Engineers, the American Institute of Mining and

Metallurgical Engineers, the American Chemical Society, the Society of Chemical Industry, the American Iron and Steel Institute, the Franklin Institute, the German Iron and Steel Institute, the American Foundrymen's Association, and the Engineers' Society of Western Pennsylvania.

Mr. Colby was elected a Member of the American Society of Civil Engineers on October 7, 1903.

ARTHUR BATEMAN CORTHELL, M. Am. Soc. C. E.*

Died May 24, 1924.

Arthur Bateman Corthell, the son of James H. and Charlotte Almy Corthell, was born on July 3, 1860, in Whitman, Mass. His father was a veteran of the Civil War.

Mr. Corthell was educated in the public schools of Providence, R. I., and entered Brown University in 1877, completing his course in March, 1881. At that time, Brown University had no comprehensive Engineering Course, but, in 1898, by a special vote of the Trustees of the University, Mr. Corthell was awarded the degree of Master of Arts.

During his college course, he was employed on vacations, and at other times, as a Draftsman for the Herreshoff Manufacturing Company of Bristol, R. I. As a result of his work for that company on the design and construction of yachts, he became an enthusiastic yachtsman and throughout his life owned or sailed yachts whenever an opportunity presented itself.

Following his graduation from Brown University, Mr. Corthell was employed for a few months as a Rodman in the City Engineer's office in Providence. In September, 1881, he left that position to take up railroad engineering as a profession, to which work he devoted the remainder of his life.

His first work was with the New York, West Shore and Buffalo Railroad Company, where he was employed from September, 1881, to December, 1884, as Assistant Engineer on Construction between Little Falls and Pittsford, as Supervisor on Maintenance, and, during the last year, as Division Engineer of that part of the line between Syracuse and Buffalo, N. Y.

Between December, 1884, and April, 1886, Mr. Corthell worked with the Knickerbocker Ice Company of New York, N. Y., making surveys and locations for the Rockland Lake Railroad and on miscellaneous engineering projects. He returned to the employ of the New York, West Shore and Buffalo Railroad Company early in 1886 and was engaged as Draftsman in the office of the Chief Engineer until the middle of that year.

In September, 1886, he entered the employ of the Fitzgerald and Mallory Construction Company and, as Principal Assistant Engineer, worked on the construction of about 300 miles of new railroads in Kansas and Colorado for the Missouri Pacific System.

From July, 1887, to March, 1888, he was employed as First Assistant Engineer on the construction of the Sioux City Bridge, over the Missouri River,

* Memoir prepared by F. C. Shepherd, M. Am. Soc. C. E.

this work carrying with it heavy pneumatic foundations with piers sunk to a depth of 92 ft.

In 1888 and 1889, Mr. Corthell served as Assistant Engineer on the construction of the Thames River Bridge and its approaches, at New London, Conn., having charge of the construction of the eastern half of the bridge foundations and the eastern approach, about 5 miles long, re-locating and designing the masonry structures.

In 1889, he became Assistant Engineer of the New York, Providence and Boston Railroad, on design and construction work and, in 1892, he was made First Assistant Engineer in charge of the design and construction of the Providence Terminal, with its approaches, being associated in this work with the late George B. Francis, M. Am. Soc. C. E.

After the completion of the Providence Terminal, Mr. Corthell followed Mr. Francis to Boston, Mass., and was employed under him from 1897 to 1899 as Principal Assistant Engineer on the construction of the South Station, one of the largest passenger terminals in the world.

On the completion of this station, he was engaged for a few months with Westinghouse, Church, Kerr and Company in Boston, returning, in February, 1900, to the South Station, where for two years he was Resident Engineer for the Boston Terminal Company, leaving there in February, 1902, to accept a position with the New York Central Railroad.

From 1902, to July, 1911, Mr. Corthell was with the New York Central Railroad, as Terminal Engineer for four years; as Assistant Executive to the Grand Central Station Architects for two years; and for the last three years as Secretary and Consulting Engineer of the Auxiliary Facilities Committee of the Grand Central Terminal.

From his association with Mr. Francis and his experience at Providence and Boston, Mr. Corthell became known as one of the leading experts in the country on terminal developments, and this undoubtedly largely influenced his selection for the position mentioned.

Under his direct supervision, the original plans of the Grand Central Terminal improvements were developed, and he was largely instrumental, not only in the preparation of the final plans, but also in the carrying out of the work toward its completion. It has been said of his relations with many associates in the planning and construction of this great terminal, that it was largely due to his efforts that the work went on harmoniously and without delay.

In 1911, Mr. Corthell became Chief Engineer of the Boston and Maine Railroad, being selected for this position because of the large construction plans contemplated, at that time, a number of which have been carried out. This position he filled until his sudden death while spending the week-end at his farm in New Boston, N. H.

He is survived by his wife, Mrs. Lena Foster Corthell, a daughter, Mrs. John T. Phelps, and two sons, Mark A. and Edwin L. Corthell, as well as by his father and mother.

As a boy, Mr. Corthell was a member of the choir of Trinity Church, Bristol, R. I., and always, during his life, was a great lover of music. He was

a member of the Protestant Episcopal Church and at the time of his death had been a Vestryman of the Church of the Epiphany, in Winchester, Mass., for several years.

He was an enthusiastic Mason, having been affiliated with Orpheus Lodge, of Providence, R. I. He was also a member of the Massachusetts Consistory.

Mr. Corthell was a man of unusual character and integrity, had a very quick, keen mind, and his ability was fully recognized in his profession, as well as by the railroad executives under whom he was employed. He was a man of many friends and one who was greatly beloved by all who ever became associated with him. His death is a distinct loss to the Engineering Profession.

He was a member of the Boston Society of Civil Engineers, the American Railway Engineering Association, the New England Railroad Club, and the Vesper Country Club of Lowell, Mass.

Mr. Corthell was elected a Member of the American Society of Civil Engineers on March 1, 1899.

WILLIAM VOORHEES JUDSON, M. Am. Soc. C. E.*

Died March 29, 1923.

William Voorhees Judson was born in Indianapolis, Ind., on February 16, 1865, and spent his boyhood in that State. His family was descended on the one side from earliest New England stock and on the other from the first Dutch settlers of New Amsterdam.

At the age of seventeen, he went to Harvard University, but while at home on a vacation during his second year he won a competitive examination for appointment to the United States Military Academy. Having been graduated third in his class in June, 1888, he was appointed a Second Lieutenant in the Corps of Engineers, U. S. Army, and served as an officer of that Corps, during the remainder of his life, reaching the grades of First Lieutenant in 1893, Captain in 1898, Major in 1906, Lieutenant-Colonel in 1912, and Colonel in 1917. During the World War he served as a Brigadier General.

On his appointment to the Corps of Engineers, Lieutenant Judson was assigned to duty with the Engineer Battalion and attended the Engineer School of Application, from which he was graduated in 1891. In the spring of that year, at her home in Lexington, Ky., he was married to Alice Carneal Clay; the marriage was characterized by unswerving mutual devotion until the day of his death.

For the next few years, Lieutenant Judson acted as Assistant Engineer to various officers in charge of river and harbor improvements on Lake Erie, on the Upper Mississippi, and at Galveston, Tex. This was followed by about 18 months as Adjutant of the only Battalion of Engineers the Army then boasted. In 1898, General Lawton requested that Captain Judson be appointed Adjutant

* Memoir prepared by William P. Wooten, M. Am. Soc. C. E., and Clay Judson, Esq., Chicago, Ill.

General of the troops which Lawton commanded. The Chief of Engineers, however, refused to let him go on the ground that he could not be spared from strictly engineering duties. This decision was a bitter disappointment to Captain Judson, and he could never quite forgive or forget it.

In 1898, he was made Recorder of the Board of Engineers and from July, 1899, to August, 1900, he served as Chief Engineer and President of the Board of Public Works of the Island of Porto Rico under the Military Government which obtained there immediately after the war with Spain. This was the most important work which he had yet done, and the excellence of his accomplishment is recognized in the Annual Report to the War Department by the Governor of Porto Rico, Gen. George W. Davis, who said of Captain Judson: "He has shown marked ability as an organizer, administrator, and as an Engineer."

Following his year in Porto Rico, Captain Judson was for 18 months in charge of river and harbor improvement and fortification construction on the Gulf Coast and in Georgia and Alabama; and for three years in Washington, D. C., as an Instructor in the Engineer School, a member of the River and Harbor Board, and Assistant to the Chief of Engineers.

In March, 1904, Captain Judson was sent as Military Observer with the Russian Army in the Russo-Japanese War. At the battle of Mukden, he was captured with a large body of Russian troops by the rapidly advancing Japanese and returned to America through Japan. His official report on the war was filled with his distinctive independence of thought and sound common sense. Some of the conclusions which he drew were not generally concurred in at that time, but their soundness was thoroughly demonstrated by the lessons of the World War.

After his return from Russia, Captain Judson was placed in charge of the River and Harbor District at Milwaukee, Wis., where he had an opportunity to put into use the concrete caissons which he had invented several years before. This device, patented and donated to the Government, was described by him in a paper* read before the Western Society of Engineers. He regarded his invention of these caissons as the most valuable single piece of construction work that he ever did for the Government. It was partially in recognition of this service that Harvard University conferred on him in 1911 the honorary degree of Master of Arts.

In March, 1909, Major Judson was appointed Engineer Commissioner of the District of Columbia. For four years he filled that office with distinction. During that period he succeeded in effecting substantial economies and improvements in the lighting service, the water distribution, the street cleaning, and other engineering activities. He designed and constructed on a large tract of land in Virginia a model work-house for 600 District prisoners. He earnestly and constantly urged the passage of legislation leading to better regulation of corporations, public utilities, and private building operations, to a termination of abuses in the insurance business, to the adoption of a definite program for the execution of public works, the acquisition of parks, and the extinction

* "Concrete-Steel Caissons: Their Development and Use for Breakwaters, Piers and Revetments," *Journal*, Western Soc. of Engrs., Vol. XIV (1909), p. 533.

of the public debt, and to the inauguration of many other measures of improvement and reform. Much of the legislation which he advocated was enacted into law during his term of office. The wisdom of many more of his suggestions has been recognized by the passage of appropriate laws during more recent years.

On the completion of his tour as Engineer Commissioner of the District of Columbia, Colonel Judson was ordered, at the special request of Gen. George W. Goethals, U. S. A. (now Major-General U. S. A. (*Retired*)), M. Am Soc. C. E., to Panama as Assistant Division Engineer of the Atlantic Division, which included the work at Gatun. Colonel Judson remained in Panama only one year, when he was relieved at his own request.

He spent the next three years in charge of the River and Harbor District and Division at Chicago, Ill., and, in 1916, was ordered to Baltimore, Md., to take charge of the river and harbor and fortification work in that District. He had been there nearly a year, when war was declared against Germany.

Colonel Judson was immediately ordered to raise, organize, and command a regiment of Engineers to be taken to France as soon as possible, but before he could begin to execute this order, it was countermanded, and he was attached to the Root Commission which was then about to start to Russia. When the Root Commission returned to America, Colonel Judson was left in Russia as Military Attaché and Chief of the American Military Mission. At about the same time he received his commission as Brigadier General in the National Army. Shortly before this he had been decorated by General Brussiloff with the order of St. Anne, 2d Class, with Swords, and a little later received from the Kerensky Government the Order of St. Stanislaus, 1st Class, with Swords.

During his stay in Russia, General Judson saw what the diplomats failed to see, that the revolutionary spirit in the Army, the lessening of the authority of the officers, and the crumbling of discipline were certain to bring chaos in their wake. As early as August 7, 1917, he cabled the War College: "At least an even chance that Russia will go out of the war within a few months. Our larger war plans should be made accordingly." He was entirely unable to convince the American and Allied diplomatic representatives of the danger, however, and they constantly reported, to December, 1917, that Russia would stay in the war to the bitter end.

In spite of the discouragement which met him at every turn, and the lack of understanding of the Ambassador, General Judson, without any official support, contributed most valuable advice and assistance to the unofficial anti-Bolshevik crusade which possibly delayed by several weeks the advent of this faction to the control of the Government. After the accession of the Bolsheviks to power, General Judson urged in vain that the inevitable be recognized and that every effort be made to guide the exercise of that power along lines most in the interest of America and the Allies. The best he was able to accomplish, however, was to obtain from the Ambassador a grudging consent to arrange for an interview with Trotsky on the eve of that official's departure to negotiate an armistice with the Germans. Trotsky was apparently moved

by General Judson's carefully prepared argument and finally said that he would demand a long armistice, the detention of enemy troops along the front, and no exchange of prisoners or products. Although he was unable to obtain these favorable terms, he did resist for months the imposition of the harsh German peace and thus delayed the utilization of the full German power on the Western Front until after the United States had become prepared to put forces into the field to help meet it. How much this delay contributed to the final victory of the Allies may be left for the careful historian to estimate.

General Judson returned to the United States in January, 1918. After a full appreciation of his work as Chief of the Military Mission in Russia, the President awarded him the Distinguished Service Medal for his services there.

During the spring and summer of 1918, he commanded the 38th Division at Hattiesburg, Miss., and was officially commended for the progress made in the training and discipline of the Division while under his command.

In the late summer of 1918, he was assigned to the command of the Port of Embarkation of New York. For the next two months he was responsible for the embarkation and transportation of 500 000 men and for the shipment of the greater part of the supplies for the Expeditionary Forces, and, in addition, was constantly responsible for the well-being of from 100 000 to 200 000 troops. The period was at the very height of the influenza epidemic which made troop movement so difficult and harrowing a task.

A slight attack of influenza and the strain under which he had been constantly working led General Judson to suspect that a physical breakdown was near. Until the day the Armistice was signed, however, he worked at high pitch, giving all his energy to the task before him. Immediately the war was over, he felt compelled to consult a physician, and it was found that his heart was greatly enlarged and that any long continuance of his life was improbable. Nevertheless, after four months in bed, although realizing that his end was always near, he resumed his task and accomplished much work of value.

Back in Chicago, a Colonel of Engineers again, in charge of river and harbor improvement, he further developed his study of that city's particular problem and worked out a plan for future harbor development in the Wolf Lake region, which he called "Illiana Harbor." The plan was outlined in some detail in a paper read before the Western Society of Engineers on November 7, 1921.*

In December, 1921, while on a trip down the Mississippi River, there came a failing of his heart more pronounced than any he had yet suffered. He fought a hard and patient battle against his fatal trouble, but was never able to recover sufficiently to resume his work, and in August, 1922, he was retired from active service. A final attack the following spring was not to be withstood, and he died on March 29, 1923, at Winter Park, Fla.

The qualities which most distinguished William Voorhees Judson were perhaps the fearless independence of his thought, the soundness of his judgment, and a never failing sense of humor. This combination of characteristics made him the best of companions, for every event took on a freshness

* "Illiana Harbor," *Journal*, Western Soc. of Engrs., Vol. XXXVII (1922), p. 201.

and new meaning when seen through his eyes. Naturally, such qualities endeared him to his many friends in all parts of the world, although they made him powerful enemies as well. Yet his independence was never either radical or careless. He had a boundless regard for history and science, and the value of his work and ideas was due to the fact that in every case they rested on the firmest foundations. He was careful to an extreme and did not believe in taking unnecessary chances; however, his conservatism, because combined with a remarkable creative faculty, led to considerable accomplishment—accomplishment which would have been greater still had he not been stricken at the very time in life when ripened experience had added so much to his productive capacity.

General Judson was elected a Member of the American Society of Civil Engineers on November 4, 1896.

JULIUS PITZMAN, M. Am. Soc. C. E.*

Died August 31, 1923.

Julius Pitzman, one of the most notable figures in the civic development of the City of St. Louis, died on August 31, 1923. His monument, more enduring than marble or granite, is the greater happiness, the larger industrial opportunity, of the city of his adoption.

He was born in Halberstadt, Prussia, June 11, 1837, the youngest of nine children. His early education was obtained at the Real Gymnasium in his native town.

Two of Mr. Pitzman's sisters, whose husbands had been strong sympathizers with the German Revolution of 1848, had been compelled to emigrate to the United States. Upon the death of his father in 1852, his mother decided to re-unite her family here. In preparation for this move, Mr. Pitzman devoted his entire time until May, 1854, to private study of English, French, and accounting.

The family arrived in New York, N. Y., after a voyage of six weeks and proceeded at once to the home of one of Mr. Pitzman's sisters near Milwaukee, Wis. Mr. Pitzman found employment as a clerk in the Post Office shortly after his arrival, but, on account of lack of opportunity for advancement, resigned the position in the fall of 1854 and moved to St. Louis, Mo., where he secured a position in a dry goods store at a salary of $8 per month. He later became connected with an agricultural implement company with an advanced compensation of $16 per month.

Commercial pursuits were not to his liking, and Mr. Pitzman entered the employ of his brother-in-law, Mr. Charles E. Salomon, who was employed as Assistant to Mr. Henry Kayser, then City Engineer of St. Louis. He applied himself with characteristic energy to the study of Mathematics and Surveying under the guidance of Mr. Salomon and Mr. Kayser, and, in 1856, was ap-

* Memoir prepared by Baxter L. Brown, W. E. Rolfe, and F. G. Jonah, Members, Am. Soc. C. E.

pointed Deputy County Surveyor of St. Louis County which, at that time, included the City of St. Louis. In 1859, he opened an office in St. Louis for the practice of Surveying. His conspicuous ability in his chosen work became speedily apparent, and he rapidly built up a lucrative business.

Mr. Pitzman's Civil War record was one of notable activity in the service of his adopted country. He volunteered in 1861 and was appointed First Lieutenant of Engineers by General Fremont, in which capacity he built Fort No. 5, located immediately west of Lafayette Park in St. Louis. Upon the completion of this work, he was transferred to the command of Col. George Thom, Chief of Topographical Engineers, on the staff of General Halleck, and under his direction, made a topographic military map of St. Louis during the winter of 1861-62.

In April, 1862, a few days after the Battle of Shiloh, he was transferred to Pittsburg Landing, Tenn., and immediately proceeded with the construction of a topographic map of the battlefield, on the completion of which he took part in the operations around Corinth, Miss.

In July, 1862, Mr. Pitzman reported to General Sherman, then commanding the Fifth Division of the Army of the Tennessee, and, during the fall of that year made the necessary surveys and military maps for the defense of Memphis, Tenn.

He entered the command of Capt. J. H. Wilson, Chief of Engineers on the staff of General Grant, in October, 1862. As the Federal Government did not, at that time, grant commissions to engineer officers, General Sherman obtained from Governor Gamble of Missouri his appointment, on November 29, 1862, as First Lieutenant, Company D, Sixth Missouri Infantry. He was then detailed for duty as Engineer and Aide de Camp and took part, until seriously wounded on May 23, 1863, in the activities preceding the Battle of Vicksburg. He was promoted on April 5, 1863, to the rank of Captain, Company D, Sixth Missouri Infantry, on the staff of the Fifteenth Army Corps.

He did not recover from the effects of his wounds until the fall of 1863, when, on account of his physical condition, he was forced to resign from active service. Shortly thereafter, he was elected County Surveyor of St. Louis County.

In September, 1864, when General Price with a large force of Confederates threatened St. Louis, Mr. Pitzman again volunteered and was appointed Major and Chief Engineer of the First Division of Missouri State Militia and took an active part in the campaign against General Price. He was mustered out of service with the Militia on November 3, 1864, and resumed his office as County Surveyor, which he held until the separation of the City of St. Louis from St. Louis County, in 1876. At this time, he was appointed a City Surveyor and held the appointment until his death on August 31, 1923.

Mr. Pitzman's vision of the future of St. Louis began to show itself with the coming of peace. To enumerate all the public work in which he was actively interested, would mean virtually to write the history of the civic development of St. Louis since the Civil War.

He developed the privately controlled residence "place" so characteristic of St. Louis, and, in the face of ultra-conservative preference for straight lines and right angles in city streets, planned and promoted many of the finest residence districts in the city, with winding roadways and beautiful vistas. Vandeventer Place, then the most exclusive residence street in the city, was laid out in 1870 on the site of Fort No. 6, built during the Civil War.

About the same time, he was active in securing the passage of ordinances providing for the widening of Jefferson and Grand Avenues, with the idea of establishing broad thoroughfares through and around the city. Neither project was carried out, because of the refusal of the City Comptroller to recommend appropriations. The city is paying to-day, and paying well, for its lack of appreciation of Mr. Pitzman's farsightedness.

Between 1872 and 1874, Mr. Pitzman represented a Citizens' Committee which advocated the lowering and bridging of the Pacific Railroad in Mill Creek Valley, which has since become the city's greatest railroad center. Differences of opinion resulted in the reference of the problem to a Board of Engineers, and this Board not only reported favorably on the Pitzman plan, but elaborated it considerably.

O'Fallon, Carondelet, and Forest Parks, large tracts of waste land far beyond the existing city limits, were secured to posterity largely through Mr. Pitzman's foresight and initiative. He led the movement which resulted in the passage of an Act of the State Legislature authorizing their acquisition. City bonds were issued for the improvement of Forest Park, and, in 1874 and 1875, this incomparable recreation ground was planned and laid out under Mr. Pitzman's supervision as Chief Engineer, in conjunction with Mr. Kern, the Landscape Architect.

He was a member of the Board of Freeholders elected in 1875, to frame a new charter for the City of St. Louis, involving an extension of the city limits. He submitted to the Board a plan contemplating the inclusion of the whole of St. Louis County. This plan was rejected by the Freeholders, and Mr. Pitzman played a large part in the final determination of the new limits. One of his main contentions was that the River des Peres, for a large portion of its course, should be included within the city, as it was inevitable that the discharge of sewage into the stream would cause a nuisance, the abatement of which would devolve on the city. The wisdom of this course has since become apparent in connection with the recently adopted plan for the improvement of the river.

While acting on the Board of Freeholders, Mr. Pitzman was largely responsible for the establishment of the Board of Public Improvements, the administrative body of the city. The Charter provided that a majority of this Board should be engineers.

He early realized that the improvement of river-front property on both sides of the Mississippi River could not be adequately completed without the establishment of definite harbor lines. In 1903, he submitted a comprehensive plan to this end, which, with slight variation, was approved by the Secretary

of War. Under this plan, the Cities of St. Louis and East St. Louis acquired possession of land valued at several million dollars.

Mr. Pitzman was mainly instrumental, during 1903 and 1904, in bringing about the establishment of the East Side Levee & Sanitary District, formed for the purpose of protecting industrial property in East St. Louis and adjacent towns in Illinois, from overflow of the Mississippi River, and to insure uninterrupted railroad communication with the East.

He was appointed, in 1901, with Mr. George Kessler, as engineer for the Louisiana Purchase Exposition, but withdrew because of differences of opinion on technical matters.

In 1917, Mr. Pitzman served as a member of a Board of Engineers appointed by the Commercial Club and Business Men's League, to make a study of industrial and commercial conditions in St. Louis. The report submitted by this Board recommended the expenditure of $62 000 000, most of which has since been approved by the citizens of St. Louis through the authorization of bond issues.

Thus, in fragmentary outline, Julius Pitzman's life work is written.

From his earliest youth to advanced age, he was continuously active in business and in contact with public affairs. He was that rare combination, a dreamer and a practical man. It was a fine tribute to a man of conspicuous ability when a prominent man of affairs in St. Louis said: "I know of no one who enjoys the confidence of his clients in higher degree than does Julius Pitzman."

On October 1, 1867, Mr. Pitzman was married, in St. Louis, to Miss Emma R. Tittman, and to them were born a daughter and two sons. Mrs. Pitzman died on October 17, 1872. On March 31, 1879, he was married to Miss Caroline Marsh Wislizenus, and to them were born four sons and one daughter.

He was an Honorary Member of the American Institute of Architects, and a member of the Engineers' Club of St. Louis, the Missouri Historical Society, the Academy of Science, the Loyal Legion of the United States, the Noonday Club, and the St. Louis Country Club.

Mr. Pitzman was elected a Member of the American Society of Civil Engineers on December 4, 1907.